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ON COMPACTNESS PROPERTIES IN THE AFFINE SOBOLEV INEQUALITY

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Abstract. The paper studies compactness properties in the affine Sobolev inequality of Gaoyong Zhang et al [6, 12]. It gives profile decompositions for sequences with bounded affine Sobolev functional, and proves existence of minimizers for model isoperimetric problems.

1. Introduction

The paper studies properties of the affine Sobolev inequality of Gaoyong Zhang et al [6, 12]

$$J_p(u) \overset{\text{def}}{=} c_{N,p}\left(\int_{S_1} \frac{\mathrm{d}S_\omega}{\|\omega\cdot\nabla u\|_p^N}\right)^{-1/N} \geq S(N,p)\|u\|_{p^*}, \tag{1.1}$$

where S_1 is a unit N-dimensional sphere, $1 \leq p < N$, $p^* = \frac{pN}{N-p}$, and $\|\cdot\|_p$ denotes the $L^p(\mathbb{R}^N)$-norm, $c_{N,p} \overset{\text{def}}{=} \left(\frac{N\omega_N\omega_{p-1}}{2\omega_{n+p-2}}\right)^{\frac{1}{p}}(N\omega_N)^{\frac{1}{N}}$, $S(N,p)$ is the best constant in the sharp Sobolev inequality, and $\omega_s \overset{\text{def}}{=} \frac{\pi^{\frac{s}{2}}}{\Gamma(1+\frac{s}{2})}$ is the area of a unit sphere in s dimensions when $s \in \mathbb{N}$.

In what follows we will use the following notations. Notation C refers to any positive constant, and its value is not fixed. Let Ω be an open set in $\mathbb{R}^N$. The space $H^{1,p}(\Omega)$ is the space of all functions $u \in L^p(\Omega)$ with weak derivatives $\nabla u \in L^p(\Omega)$, endowed with the norm $(\|u\|_p^p + \|\nabla u\|_p^p)^{1/p}$. Its subspace defined as a closure of $C_0^\infty(\Omega)$ is denoted as $H_0^{1,p}(\Omega)$. The completion of $C_0^\infty(\Omega)$ in the norm $\|\nabla u\|_p$ is denoted as $\dot{H}^{1,p}(\Omega)$. It is identified as a space of functions by an embedding, if $N > p$, or if Ω is bounded, and in the latter case it coincides with $H_0^{1,p}(\Omega)$. We consider the functional J_p as defined on $\dot{H}^{1,p}(\mathbb{R}^N)$.

Affine Sobolev inequality is a refinement of the limiting Sobolev inequality $\|\nabla u\|_p \geq S(N,p)\|u\|_{p^*}$ in the sense that J_p is bounded by the gradient norm $\|\nabla\cdot\|_p$ (inequality (7.1) in [6] that easily follows from the definition), but not vice versa. Functionals $\|\nabla\cdot\|_p$ and J_p are invariant with respect to actions of translations, dilations, and orthogonal rotations. Furthermore, they coincide on radially symmetric functions (once the normalization constant $c_{N,p}$ is chosen as above). On the other

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hand, unlike the gradient norm, the affine Sobolev functional is invariant with respect to the action of the group $SL(N)$, i.e. $J_p(u\circ T) = J_p(u)$ whenever $\det T = 1$. This easily follows from the following representation of the affine Sobolev functional ([6, p.20], easily derived by radial integration):

$$J_p(u) = c_{N,p}\left(\frac{1}{(N-1)!}\int_{\mathbb{R}^N} e^{-\|\xi\cdot\nabla u\|_p}\mathrm{d}\xi\right)^{-1/N}. \tag{1.2}$$

Since $\sup_{\det T=1}\|u\circ T\|_{\dot H^{1,p}} = \infty$ for any $u\in C_0^\infty(\mathbb{R}^N)\setminus\{0\}$, as it can be easily tested on diagonal matrices, the inequality $\|\nabla u\|_p \le CJ_p(u)$ does not hold.

While the Sobolev inequality defines a continuous embedding $\dot H^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, where $\dot H^{1,p}(\mathbb{R}^N)$ is a completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the gradient norm $\|\nabla u\|_p$, the functional J_p is not a norm. Indeed, evaluating it on a sequence $u + v\circ T_k$ where u and $v\circ T_k$ have disjoint supports, $T_k\in SL(N)$ and $|T_k|\to\infty$, one can easily see from Lemma 4.3 below that it does not satisfy the triangle inequality.

Applications of the affine Sobolev inequality to information theory are discussed in [6]. For $N\le p$, analogs of Moser-Trudinger and Morrey-Sobolev inequalities involving the functional J_p have been studied in [3].

The main objective of this paper is to generalize properties of the affine Sobolev inequality, studied in [8] for the case $p=2$, to all $p>1$. In particular, we prove that, similarly to the Sobolev embeddings, the set $\{u\in\dot H^{1,p}(\Omega), J_p(u)\le 1\}$ with bounded Ω is compact in $L^q(\Omega)$, $1\le q<p^*$, and give profile decompositions for sequences with bounded J_p when Ω is not necessarily bounded. In the case $p=2$ many arguments in [8] resort to an identity

$$J_2(u) = \inf_{T\in SL(N)}\|\nabla u\circ T\|_2, \tag{1.3}$$

known only for $p=2$. For general p only a weaker relation is known ([5, Theorem 1.2]):

$$C'\min_{T\in SL(N)}\|\nabla(u\circ T)\|_p \le J_p(u) \le C\min_{T\in SL(N)}\|\nabla(u\circ T)\|_p,\ u\in\dot H^{1,p}(\mathbb{R}^N). \tag{1.4}$$

(Note that the minimum in (1.4) is attained since the map $T\mapsto\|\nabla(u\circ T)\|_p$ is continuous on $SL(N)$, and its closed sublevel sets are compact on $SL(N)$, coercive behavior of this map can be inferred from the values of the trace of T^*T).

Remark 1.1. Note that from (1.4) it easily follows that the Friedrichs inequality $\|\nabla u\|_p \ge C\|u\|_p$, $u\in H_0^{1,p}(\Omega)$, holds for an open set $\Omega\subset\mathbb{R}^N$ if and only if its affine counterpart $J_p(u)\ge C\|u\|_p$, $u\in H_0^{1,p}(\Omega)$, holds true.

Since we do not know if one can identify $J_p(u)$ as a scalar factor of $\min_{T\in SL(N)}\|\nabla(u\circ T)\|_p$ when $p\ne 2$, if one defines the affine p-Laplacian as the Fréchet derivative of J_p, it may be different from the Fréchet derivative of $\inf_{T\in SL(N)}\|\nabla(u\circ T)\|_p$. Consequently, treatment of variational problems involving J_p in this paper is different from that in [8] when $p=2$.

The affine Sobolev inequality (1.1) easily follows of course from the usual Sobolev inequality and (1.4):

$$\|u\|_{p^*} \le C \inf_{T\in SL(N)} \|u\circ T\|_{p^*} \le C \inf_{T\in SL(N)} \|\nabla(u\circ T)\|_p \le CJ_p(u). \tag{1.5}$$

Note also that from the left hand side of (1.4) it follows that if $J_p(u) = 0$, $u \in \dot{H}^{1,p}(\Omega)$, and $\Omega \subset \mathbb{R}^N$ is a convex domain, then there is a family of parallel hyperplanes, such that u is constant on their intersection with Ω.

In Section 2 we prove two Hardy-type inequalities involving the affine Sobolev functional. In Section 3 we study compactness properties of J_p. In Section 4 we give profile decompositions for sequences with a J_p-bound. Section 5 gives existence of minimizers to some isoperimetric problems involving J_p. The Appendix contains, for the convenience of the reader, profile decompositions for $H^{1,p}(\mathbb{R}^N)$ and $\dot{H}^{1,p}(\mathbb{R}^N)$.

2. Inequalities of Hardy type for affine Sobolev functional

Theorem 2.1. *Let $1 \le p < N$. Then for any $u \in \dot{H}^{1,p}(\mathbb{R}^N)$,*

$$C_H \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p}\mathrm{d}x \le J_p(u)^p, \tag{2.1}$$

where $C_H = \left(\frac{N-p}{p}\right)^p$ is the best constant in the Hardy inequality as well as in the inequality above.

Proof. Let $v^\star$ denote a symmetric decreasing rearrangement of v. Fix $u \in \dot{H}^{1,p}(\mathbb{R}^N)$, and let $T_u \in SL(N)$ be a minimizer in $\|\nabla(u\circ T)\|_p$. Applying Littlewood-Hardy inequality at the first step, Hardy inequality at the second step, equality of J_p and the gradient norm on the third step, and Polya-Szegö inequality for J_p from [6], (see also [3, Theorem 2.1]) at the last step, we have

$$C_H \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p}\mathrm{d}x \le C_H \int_{\mathbb{R}^N} \frac{|u^\star(x)|^p}{|x|^p}\mathrm{d}x \le \|\nabla u^\star\|_p^p = J_p(u^\star)^p \le J_p(u)^p.$$

The constant C_H is the largest possible since it is optimal for radial functions. Indeed, in such a case, $J_p = \|\nabla\cdot\|_p$ (see [6, Sction 7]) and so the above inequality reverts to the standard one. □

In what follows $B_r(x)$ will denote an open Euclidean ball of radius r centered at $x \in \mathbb{R}^N$. Let

$$\mathcal{E}_p(u) \stackrel{\text{def}}{=} \inf_{T\in SL(N)} \|\nabla(u\circ T)\|_p^p. \tag{2.2}$$

Theorem 2.2. *There exists $C_N > 0$ such that for any $u \in \dot{H}^{1,N}(B_1(0))$,*

$$\left(\frac{N-1}{N}\right)^N \int_{B_1(0)} \frac{|u(x)|^N}{(|x|\log\frac{1}{|x|})^N}\mathrm{d}x \le \mathcal{E}_N(u) \le C_N J_N(u)^N. \tag{2.3}$$

Proof. If we consider the unit ball $B_1(0)$ as the Poincaré ball model of the hyperbolic space $\mathbb{H}^N$, then $\|\nabla u\|_N$ coincides with the invariant norm $\|du\|_N$ of the Sobolev space $H^{1,N}(\mathbb{H}^N)$. Let $v^\star$ denote a symmetric decreasing rearrangement of a function v with respect to the Riemannian measure of $\mathbb{H}^N$, which in the Ponicaré ball model has the form $\frac{2^N \mathrm{d}x}{(1-|x|^2)^N}$. Let $u \in H^{1,N}(\mathbb{H}^N)$. Applying the Hardy-Littlewood inequality for rearrangements for the hyperbolic space (see [2]) at the first step (noting that $\left(\frac{1-r^2}{r\log\frac{1}{r}}\right)^N$ is a decreasing function), Leray inequality for the N-Laplacian (see [1])

$$\left(\frac{N-1}{N}\right)^N \int_{B_1(0)} \frac{|u(x)|^N}{(|x|\log\frac{1}{|x|})^N}\mathrm{d}x \le \|\nabla u\|_N^N, \tag{2.4}$$

at the second step, using the measure-preserving property of $SL(N)$ at the third step, Polya-Szegö inequality for hyperbolic space ([2]) at the fourth step, and (1.4) at the last step, we have, for any $T \in SL(N)$,

$$\begin{gathered}\left(\frac{N-1}{N}\right)^N \int_{B_1(0)} \frac{|u(x)|^N}{(|x|\log\frac{1}{|x|})^N}\mathrm{d}x \le \\ \left(\frac{N-1}{N}\right)^N \int_{B_1(0)} \frac{|u^\star(x)|^N}{(|x|\log\frac{1}{|x|})^N}\mathrm{d}x \le \\ \|\nabla u^\star\|_N^N = \|\nabla(u\circ T)^\star\|_N^N \le \|\nabla(u\circ T)\|_N^N.\end{gathered}$$

Taking the infimum over $T \in SL(N)$ and using (1.4) we arrive at (2.3). $\square$

3. Affine-null and affine-flask domains. Compactness in L^q.

In what follows $|\Omega|$ will denote the Lebesgue measure of a set. Recall the definition of the lower limit for a sequence (X_k) of sets:

$$\liminf X_k \overset{\text{def}}{=} \bigcup_{n\in\mathbb{N}} \bigcap_{k\ge n} X_k.$$

The following definition can be used as a sufficient condition on Ω, for compactness of the Sobolev embedding $H^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $p < q < p^*$ even when Ω has infinite measure (see [11]).

Definition 3.1. A subset Ω of $\mathbb{R}^N$ will be called a shifts-null set if for any sequence $(y_k) \subset \mathbb{Z}^N$, such that $|y_k| \to \infty$,

$$|\liminf(\Omega - y_k)| = 0. \tag{3.1}$$

Definition 3.2. A subset Ω of $\mathbb{R}^N$ will be called an affine-null set if for any sequences $(T_k) \subset SL(N)$ and $(y_k) \subset \mathbb{Z}^N$, such that $|T_k| + |y_k| \to \infty$,

$$|\liminf T_k^{-1}(\Omega - y_k)| = 0. \tag{3.2}$$

Note that any bounded set is affine-null. An example of an unbounded affine null set is $\{(x_1, \bar{x}) \in \mathbb{R}\times\mathbb{R}^{N-1} : |\bar{x}| < e^{-x_1^2}\}$. Not every null set relative to the group of shifts alone (i.e. $\forall (y_k) \subset \mathbb{R}^N$ $|\liminf(\Omega - y_k)| = 0$) is affine-null. In particular, the set $\{(x_1, \bar{x}) \in \mathbb{R}\times\mathbb{R}^{N-1} : |\bar{x}| < (1+\log(1+|x_1|))^{-1}\}$ is shifts-null but not affine-null.

Definition 3.3. An open subset Ω of $\mathbb{R}^N$ will be called affine-flask set if for any $(T_k) \subset SL(N)$ and $(y_k) \subset \mathbb{Z}^N$, such that $|y_k| + |T_k| \to \infty$, there exist a $y \in \mathbb{Z}^N$ and a $T \in SL(N)$ such that

$$\left|\liminf T_k^{-1}(\Omega - y_k) \setminus (T\Omega + y)\right| = 0. \tag{3.3}$$

In other words, $\liminf T_k^{-1}(\Omega - y_k)$ is contained in the image of Ω modulo an affine transformation and up to a set of zero measure. Obviously an affine-null set as well as $\mathbb{R}^N$ are affine flask sets. The union of unit balls $\bigcup_{n\in\mathbb{N}} B_1(n^4 e_0)$, $|e_0| = 1$, is an affine flask set. If one connects consecutive balls by circular cylinders of corresponding radius e^{-n} that have $\mathbb{R}e_0$ as their common axis, one gets a connected affine flask set. On the other hand a cylindrical domain with a smooth boundary is an affine flask set only if it is $\mathbb{R}^N$. Indeed, let $\Omega = \mathbb{R} \times \omega$ and let T_k be a diagonal matrix with diagonal entries $k^{1-N}, k, \ldots, k$. Then $\liminf T_k\Omega = \mathbb{R}^N$.

Theorem 3.4. *F Let $\Omega \subset \mathbb{R}^N$ be an affine-null domain [for example, a bounded domain]. Then the set $\mathcal{B} = \{u \in H_0^{1,p}(\Omega);\ J_p(u) \le 1\}$ is relatively compact in $L^q(\Omega)$, $1 < p < q < p^*$. If $|\Omega| < \infty$, this is true also for $q \in [1, p]$.*

Note that the set $\mathcal{B}$ is not bounded in $H_0^{1,p}(\Omega)$.

Proof. Let $(u_k) \subset \mathcal{B}$ and consider it as a sequence in $H^{1,p}(\mathbb{R}^N)$. Let $T_k \in SL(N)$ correspond to the minima in (1.4). Let $v_k = u_k \circ T_k$. Then (v_k) is a bounded sequence in $H_0^{1,p}(\Omega)$, which we will consider as a sequence in $H^{1,p}(\mathbb{R}^N)$. If $|T_k| \to \infty$ then by (3.2), $v_k(\cdot - y_k) \rightharpoonup 0$ in $H^{1,p}(\mathbb{R}^N)$ for any sequence $(y_k) \subset \mathbb{R}^N$ (for details see the argument in the proof in [11, Lemma 4.1]), which implies (e.g. by Proposition 6.2) that $v_k \to 0$ in L^q, $p < q < p^*$, and thus $u_k \to 0$ in L^q. Otherwise, there is a renamed subsequence of (T_k) convergent to some $T \in SL(N)$. Passing again to a renamed weakly convergent subsequence we may assume that $v_k \rightharpoonup v$ in $H^{1,p}(\mathbb{R}^N)$, and thus $u_k \rightharpoonup v \circ T^{-1}$ in $H_0^{1,p}(\Omega)$. On the other hand, from (3.2) we can infer that for any sequence $(y_k) \subset \mathbb{R}^N$, $(v_k - v)(\cdot - y_k) \rightharpoonup 0$ in $H^{1,p}(\mathbb{R}^N)$ and thus, setting $u \stackrel{\text{def}}{=} v \circ T^{-1}$, $\|u_k - u\|_q \le \|v_k - v\|_q + \|u \circ T - u \circ T_k\|_q \to 0$. $\square$

4. Profile decompositions

In this section we outline concentration behavior of sequences with bounded values of J_p (note that they are not necessarily bounded in the Sobolev norm). From now on we assume that $p > 1$.

Theorem 4.1. *Let $(u_k) \subset \dot{H}^{1,p}(\mathbb{R}^N)$, $1 < p < N$, satisfy $J_p(u_k) \le C$ with some $C > 0$. There exist $(T_k) \subset SL(N)$, $w^{(n)} \in \dot{H}^{1,p}(\mathbb{R}^N)$, $(y_k^{(n)})_{k\in\mathbb{N}} \subset \mathbb{R}^N$, $(j_k^{(n)})_{k\in\mathbb{N}} \subset \mathbb{Z}$ with $n \in \mathbb{N}$, and disjoint sets $\mathbb{N}_0, \mathbb{N}_{+\infty}, \mathbb{N}_{-\infty} \subset \mathbb{N}$, $\mathbb{N}_0 \cup \mathbb{N}_{+\infty} \cup \mathbb{N}_{-\infty} = \mathbb{N}$, such that, for a renumbered subsequence of (u_k),*

$$2^{-\frac{N-p}{p} j_k^{(n)}} u_k(T_k(2^{-j_k^{(n)}} \cdot + y_k^{(n)})) \rightharpoonup w^{(n)},\ n \in \mathbb{N}, \tag{4.1}$$

$$|j_k^{(n)} - j_k^{(m)}| + |2^{j_k^{(n)}}(y_k^{(n)} - y_k^{(m)})| \to \infty \text{ for } n \ne m, \tag{4.2}$$

$$\sum_{n\in\mathbb{N}} \|\nabla w^{(n)}\|_p^p \le C \liminf J_p(u_k)^p, \tag{4.3}$$

$$u_k - \left[\sum_{n\in\mathbb{N}} 2^{\frac{N-p}{p}j_k^{(n)}} w^{(n)}(2^{j_k^{(n)}}(\cdot - y_k^{(n)}))\right] \circ T_k^{-1} \to 0 \textit{ in } L^{p^*}, \tag{4.4}$$

and the series in the square brackets above converges in $\dot{H}^{1,p}(\mathbb{R}^N)$ *unconditionally in* n *and uniformly with respect to* k.

Moreover, $1 \in \mathbb{N}_0$, $y_k^{(1)} = 0$; $j_k^{(n)} = 0$ *whenever* $n \in \mathbb{N}_0$; $j_k^{(n)} \to -\infty$ *(resp.* $j_k^{(n)} \to +\infty$*) whenever* $n \in \mathbb{N}_{-\infty}$ *(resp.* $n \in \mathbb{N}_{+\infty}$*); and* $y_k^{(n)} = 0$ *whenever* $|2^{j_k^{(n)}} y_k^{(n)}|$ *is bounded.*

Proof. Let $T_k \in SL(N)$ be those that realize the minimum in the left hand side of (1.4). Let $v_k = u_k \circ T_k$ and apply Theorem 6.1 from Appendix. To conclude the proof of Theorem 4.1 it remains to note that (6.4) gives (4.4) by composing the left and the right hand side with T_k^{-1} on the right, and that the right hand side of (6.3) yields the right hand side of (4.3) by (1.4). $\square$

An analogous decomposition for sequences with bounded $J_p^p + \|\cdot\|_p^p$ can be derived in a completely analogous way from Proposition 6.2 in Appendix:

Proposition 4.2. *Let* $(u_k) \subset H^{1,p}(\mathbb{R}^N)$, $1 < p < N$, *be a sequence such that* $J_p(u_k)^p + \|u_k\|_p^p \le C$ *with some* $C > 0$. *There exist* $w^{(n)} \in H^{1,p}(\mathbb{R}^N)$, $(T_k) \subset SL(N)$, *and* $(y_k^{(n)})_{k\in\mathbb{N}} \subset \mathbb{Z}^N$, $y_k^{(1)} = 0$, $n \in \mathbb{N}$, *such that, on a renumbered subsequence,*

$$u_k(T_k(\cdot + y_k^{(n)})) \rightharpoonup w^{(n)}, \tag{4.5}$$

$$|y_k^{(n)} - y_k^{(m)}| \to \infty \textit{ for } n \neq m, \tag{4.6}$$

$$\sum_{n\in\mathbb{N}} \|w^{(n)}\|_{H^{1,p}}^p \le C \liminf J_p(u_k)^p + \|u_k\|_p^p, \tag{4.7}$$

$$u_k - \left[\sum_{n\in\mathbb{N}} w^{(n)}(\cdot - y_k^{(n)})\right] \circ T_k^{-1} \to 0 \textit{ in } L^q(\mathbb{R}^N), q \in (p, p^*), \tag{4.8}$$

and the series in the square brackets above converges in $H^{1,p}(\mathbb{R}^N)$ *unconditionally in* n *and uniformly with respect to* k.

Lemma 4.3. *If* $\alpha > 1$, *then the map*

$$\Phi_\alpha(u) \overset{\text{def}}{=} \left(\int_{S_1} \frac{\mathrm{d}S_\omega}{u(\omega)^\alpha}\right)^{-1/\alpha}, \tag{4.9}$$

positively homogeneous of degree 1, is concave on the set of positive continuous functions on S_1, *or, equivalently, satisfies* $\Phi_\alpha(u_1 + u_2) \ge \Phi_\alpha(u_1) + \Phi_\alpha(u_2)$.

Proof. We infer concavity from calculation of the second derivative. Let v be a continuous function on S_1. Then

$$\Phi_\alpha'(u)[v] = \left(\int_{S_1} \frac{\mathrm{d}S_\omega}{u(\omega)^\alpha}\right)^{-\frac{1}{\alpha}-1} \int_{S_1} \frac{v(\omega)dS_\omega}{u(\omega)^{\alpha+1}},$$

and

$$\Phi_\alpha''(u)[v,v] =$$

$$(-\frac{1}{\alpha}-1)(-\alpha)\left(\int_{S_1}\frac{\mathrm{d}S_\omega}{u(\omega)^\alpha}\right)^{-\frac{1}{\alpha}-2}\left(\int_{S_1}\frac{v(\omega)\mathrm{d}S_\omega}{u(\omega)^{\alpha+1}}\right)^2-$$
$$(\alpha+1)\int_{S_1}\frac{v(\omega)^2\mathrm{d}S_\omega}{u(\omega)^{\alpha+2}}\left(\int_{S_1}\frac{\mathrm{d}S_\omega}{u(\omega)^\alpha}\right)^{-\frac{1}{\alpha}-1}.$$

Applying the Cauchy inequality to the integral $\int_{S_1}\frac{v(\omega)\mathrm{d}S_\omega}{u(\omega)^{\alpha+1}}$ understood as the scalar product of functions 1 and $\frac{v}{u}$ in $L^2(S_1;\frac{\mathrm{d}S_\omega}{u(\omega)^\alpha})$, we get that $\Phi_\alpha''(u)[v,v]\le 0$. ☐

Proposition 4.4. *Assume conditions of Theorem 4.1 and let $w^{(n)}$, $n\in\mathbb{N}$ be as in the theorem. Then*

$$\sum_{n\in\mathbb{N}} J_p(w^{(n)})^p \le \liminf J_p(u_k)^p \tag{4.10}$$

and

$$\|u_k\|_{p^*}^{p^*}\to\sum_{n\in\mathbb{N}}\|w^{(n)}\|_{p^*}^{p^*}. \tag{4.11}$$

Proof. Note that for any $\omega\in S_1$,

$$\sum_{n\in\mathbb{N}}\|\omega\cdot\nabla w^{(n)}\|_p^p\le\liminf\|\omega\cdot\nabla u_k\|_p^p \tag{4.12}$$

(for details see, for example, the argument in [9] for the inequality $\sum_{n\in\mathbb{N}}\|\nabla w^{(n)}\|_p^p\le \liminf\|\nabla u_k\|_p^p$). This together with Lemma 4.3 for $\alpha=N/p$ gives relation (4.10). Relation (4.11) is an iteration of the classical Brezis-Lieb lemma based on isometry of the scaling transformations (see e.g [4]). ☐

Remark 4.5. Note that under conditions of Proposition 4.2, iteration of the Brezis-Lieb lemma (cf. [4]) gives

$$\|u_k\|_q^q\to\sum_{n\in\mathbb{N}}\|w^{(n)}\|_q^q \text{ for any } q\in(p,p^*). \tag{4.13}$$

5. Some variational problems

Theorem 5.1. *Let $\Omega\subset\mathbb{R}^N$ be an affine-null domain [for example, a bounded domain] with a piecewise-C^1-boundary. Then the minimum in the problem*

$$\kappa_{p,q}=\inf_{u\in H_0^{1,p}(\Omega),\|u\|_q=1} J_p(u),\ p<q<p^*, \tag{5.1}$$

is attained. If $|\Omega|<\infty$, this is also true whenever $q\in[1,p]$.

Proof. Let $(u_k)\subset H_0^{1,p}(\Omega)$ be a minimizing sequence. Consider it as a sequence in $H^{1,p}(\mathbb{R}^N)$. Let $T_k\in SL(N)$ correspond to the minima in (1.4). Repeating the argument in the proof of Theorem 3.4, we may assume, for a suitable renamed subsequence, that either $|T_k|\to\infty$ and then $u_k\to 0$ in L^q, or $T_k\to T\in SL(N)$, and (u_k) converges weakly in $H_0^{1,p}(\Omega)$ as well as in $L^q(\Omega)$ to some u. The former case is ruled out, since by assumption $\|u_k\|_q=1$. In the latter case, lower semicontinuity of the norm implies that $\|\nabla u\|_p\le\kappa_{p,q}$. Then by (1.4) $J_p(u)\le\kappa_{p,q}$, and thus u is necessarily a minimizer. ☐

Theorem 5.2. *Let $q \in (p, p^*)$ and let $\Omega \subset \mathbb{R}^N$ be an open affine flask set with a piecewise-C^1 boundary [for example, $\Omega = \mathbb{R}^N$]. Then the minimum in the problem*

$$\kappa = \inf_{u \in H_0^{1,p}(\Omega): \|u\|_q = 1} J_p(u)^p + \|u\|_p^p \tag{5.2}$$

is attained.

Proof. Let $(u_k) \subset H_0^{1,p}(\Omega)$ be a minimizing sequence. Consider it as a sequence in $H^{1,p}(\mathbb{R}^N)$. Let $(T_k) \subset SL(N)$ and let $w^{(n)}$, $n \in \mathbb{N}$, be as in Theorem 4.2, so we have $\|\nabla(u_k \circ T_k)\|_p \leq C J_p(u_k \circ T_k) \leq C$. From the iterated Brezis-Lieb Lemma we have

$$1 = \|u_k\|_q^q = \sum_n \|w^{(n)}\|_q^q. \tag{5.3}$$

Let $t_n = \|w^{(n)}\|_q^q$.

By (4.10) and iterated Brezis-Lieb lemma,

$$\begin{aligned} \kappa = & \quad \lim J_p(u_k(T_k \cdot - y + y_k^{(n)}))^p + \|u_k(T_k \cdot - y + y_k^{(n)})\|_p^p \\ \geq & \quad \sum_{n \in \mathbb{N}} J_p(w^{(n)})^p + \|w^{(n)}\|_p^p. \end{aligned} \tag{5.4}$$

Since Ω is an affine-flask set, (3.3) and (4.5) imply that with some $T^{(n)} \in SL(N)$ and some $y_n \in \mathbb{R}^N$ one has

$$u_k(T_k((T^{(n)})^{-1} \cdot - y_n) + y_k^{(n)}) \rightharpoonup w^{(n)}((T^{(n)})^{-1}(\cdot - y_n)) \in H_0^{1,p}(\Omega).$$

From (5.4) we have

$$\kappa \geq \sum_{n \in \mathbb{N}} \kappa t_n^{p/q}, \tag{5.5}$$

which, since by (5.3) $\sum_n \|w^{(n)}\|_q^q = 1$, can hold only if $t_n = 0$ for $n \neq m$ and $t_m = 1$ with some $m \in \mathbb{N}$. Consequently $w^{(m)}((T^{(m)})^{-1}(\cdot - y_m))$ is a minimizer. □

Theorem 5.3. *Let $q \in (p, p^*)$ and let $V \in L^\infty(\mathbb{R}^N)$ satisfy $\lim_{|x| \to \infty} V(x) = 1$ and $V(x) < 1$. Then the minimum in the problem*

$$\kappa' = \inf_{u \in H^{1,p}(\mathbb{R}^N), \|u\|_q = 1} J_p(u)^p + \int_{\mathbb{R}^N} V(x)|u(x)|^p \mathrm{d}x \tag{5.6}$$

is attained.

Proof. Let $(u_k) \subset C_0^\infty(\mathbb{R}^N)$ be a minimizing sequence. Let $(T_k) \subset SL(N)$ and let $w^{(n)}$, $n \in \mathbb{N}$, be as in Proposition 4.2. From the iterated Brezis-Lieb Lemma we have

$$1 = \|u_k\|_q^q = \sum_n \|w^{(n)}\|_q^q. \tag{5.7}$$

Let $t_n = \|w^{(n)}\|_q^q$.

Let us represent $J_p(u_k \circ T_k)^p + \int V(x)|u_k(x)|^p \mathrm{d}x$ as $J_p(u_k \circ T_k)^p + \|u_k \circ T_k\|_p^p + \int (V(x) - 1)|u_k(T_k x)|^p \mathrm{d}x$ and note that the last term is weakly continuous in $H^{1,p}(\mathbb{R}^N)$.

Assume first that $|T_k| \to \infty$. Then by (4.10) we have

$$\kappa' = \quad \lim J_p(u_k(T_k \cdot - y + y_k^{(n)}))^p + \|u_k\|_p^p$$

$$\geq \sum_{n\in\mathbb{N}} J_p(w^{(n)})^p + \|w^{(n)}\|_p^p$$

$$\geq \sum_{n\in\mathbb{N}} \kappa t_n^{p/q} \geq \kappa, \tag{5.8}$$

where κ is as in (5.2). Here the last but one inequality follows by (5.2) since $\frac{w}{\|w\|_q}$ is a unit vector in L^q, while the last inequality follows since $\sum_n t_n = 1$ and $p < q$.

Evaluation of the left hand side of (5.6) at the minimizer of (5.2) gives, however, that $\kappa' < \kappa$, which is a contradiction. Consequently, on a suitable renamed subsequence, we have $T_k \to T \in SL(N)$. In this case $u_k \rightharpoonup w^{(1)} \circ T^{-1}$ and (4.10) gives

$$\kappa' = \lim J_p(u_k \circ T_k)^p + \|u_k\|_p^p + \int (V(x)-1)|w^{(1)} \circ T^{-1}|^p \mathrm{d}x$$

$$\geq \kappa' t_1^{p/q} + \sum_{n=2}^{\infty} \kappa t_n^{p/q}, \tag{5.9}$$

which is false unless $t_n = 0$ for $n > 1$ and $t_1 = 1$. Consequently $w^{(1)} \circ T^{-1}$ is a minimizer. ☐

6. Appendix

The following theorem is a trivial refinement of [9, Theorem 2] (Sergio Solimini).

Theorem 6.1. *Let $(v_k) \subset \dot{H}^{1,p}(\mathbb{R}^N)$, $N > p > 1$, be a bounded sequence. There exist $w^{(n)} \in \dot{H}^{1,p}(\mathbb{R}^N)$, $(y_k^{(n)})_{k\in\mathbb{N}} \subset \mathbb{R}^N$, $(j_k^{(n)})_{k\in\mathbb{N}} \subset \mathbb{Z}$ with $n \in \mathbb{N}$, and disjoint sets $\mathbb{N}_0, \mathbb{N}_{+\infty}, \mathbb{N}_{-\infty} \subset \mathbb{N}$, $\mathbb{N}_0 \cup \mathbb{N}_{+\infty} \cup \mathbb{N}_{-\infty} = \mathbb{N}$, such that, for a renumbered subsequence of (v_k),*

$$2^{-\frac{N-p}{p} j_k^{(n)}} v_k(2^{-j_k^{(n)}} \cdot + y_k^{(n)}) \rightharpoonup w^{(n)},\ n \in \mathbb{N}, \tag{6.1}$$

$$|j_k^{(n)} - j_k^{(m)}| + |2^{j_k^{(n)}}(y_k^{(n)} - y_k^{(m)})| \to \infty \text{ for } n \neq m, \tag{6.2}$$

$$\sum_{n\in\mathbb{N}} \|\nabla w^{(n)}\|_p^p \leq \limsup \|\nabla v_k\|_p^p, \tag{6.3}$$

$$v_k - \sum_{n\in\mathbb{N}} 2^{\frac{N-p}{p} j_k^{(n)}} w^{(n)}(2^{j_k^{(n)}}(\cdot - y_k^{(n)})) \to 0 \text{ in } L^{p^*}(\mathbb{R}^N), \tag{6.4}$$

and the series above converges in $\dot{H}^{1,2}(\mathbb{R}^N)$ unconditionally and uniformly with respect to k.

Moreover, $1 \in \mathbb{N}_0$, $y_k^{(1)} = 0$; $j_k^{(n)} = 0$ whenever $n \in \mathbb{N}_0$; $j_k^{(n)} \to -\infty$ (resp. $j_k^{(n)} \to +\infty$) whenever $n \in \mathbb{N}_{-\infty}$ (resp. $n \in \mathbb{N}_{+\infty}$); and $y_k^{(n)} = 0$ whenever $|2^{j_k^{(n)}} y_k^{(n)}|$ is bounded.

Note that the unconditional convergence of the series is not stated in the original version of the theorem, but can be easily inferred from the proof. This omission has been remedied in the Banach space version of [10, Theorem 2.6]. This remark applies also to the easy corollary below.

Proposition 6.2. *Let $(u_k) \subset H^{1,2}(\mathbb{R}^N)$ be a bounded sequence. There exist $w^{(n)} \in H$, $(y_k^{(n)})_{k\in\mathbb{N}} \subset \mathbb{Z}^N$, $y_k^{(1)} = 0$, with $n \in \mathbb{N}$, such that, on a renumbered subsequence,*

$$u_k(\cdot + y_k^{(n)}) \rightharpoonup w^{(n)}, \tag{6.5}$$

$$|y_k^{(n)} - y_k^{(m)}| \to \infty \text{ for } n \neq m, \tag{6.6}$$

$$\sum_{n\in\mathbb{N}} \|w^{(n)}\|^p_{H^{1,p}} \leq \limsup \|u_k\|^p_{H^{1,p}}, \tag{6.7}$$

$$u_k - \sum_{n\in\mathbb{N}} w^{(n)}(\cdot - y_k^{(n)}) \to 0 \text{ in } L^q(\mathbb{R}^N), q \in (p, p^*), \tag{6.8}$$

and the series in (6.8) converges in $H^{1,q}(\mathbb{R}^N)$ unconditionally in n and uniformly in k.

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