INVARIANT GRAPHS AND SPECTRAL TYPE OF SCHRÖDINGER OPERATORS

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ABSTRACT. In this paper we study spectral properties of Schrödinger operators with quasi-periodic potentials related to quasi-periodic action minimizing trajectories for analytic twist maps. We prove that the spectrum contains a component of absolutely continuous spectrum provided that the corresponding trajectory of the twist map belongs to an analytic invariant curve.

1. INTRODUCTION

In this paper we discuss connections between the Aubry-Mather theory and the spectral theory of Schrödinger operators with quasi-periodic potentials. The Aubry-Mather theory was developed in the 1980s. In a certain sense it can be considered as a global extension of the KAM theory. It also provides a variational approach to the KAM phenomenon. In what follows we discuss only the 2D twist map setting corresponding to Hamiltonian systems with two degrees of freedom. Namely, let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and consider a family $\{\psi_{\lambda f}\}_{\lambda \in \mathbb{R}}$ of standard type maps on the cylinder:

$$\psi_{\lambda f} \colon \begin{cases} \mathbb{T} \times \mathbb{R} & \to & \mathbb{T} \times \mathbb{R}, \\ (\varphi, r) & \mapsto & (\varphi + r + \lambda f(\varphi) \bmod 1, r + \lambda f(\varphi)), \end{cases}$$

where f is a smooth 1-periodic function with mean value zero and λ is a coupling constant. The main object of interest is the set of quasi-periodic trajectories for the map $\psi_{\lambda f}$ for a fixed irrational rotation number α. One can show that such trajectories always exist. Moreover, for some $a = a(\alpha) \in \mathbb{R}$, they correspond to minimizers for the Lagrangian action $A_{\lambda f, \alpha}$:

$$A_{\lambda f, \alpha}((u_n)_{n \in \mathbb{Z}}) := \frac{1}{2} \sum_{n \in \mathbb{Z}} (u_{n+1} - u_n - a)^2 + \lambda \sum_{n \in \mathbb{Z}} F(u_n),$$

where $F' = f$. However the behavior of such minimizers depends on the coupling constant λ. For small values of $|\lambda|$ the set of minimizers belongs to a smooth invariant curve, and under certain arithmetic conditions, the dynamics on this curve is conjugated to the rigid rotation by angle α. On the other hand, for large values of $|\lambda|$ minimizers form disjoint Cantor-type sets, sometimes called *cantori*. The theory describing properties of such invariant sets is often called the weak KAM theory.

It is believed that one has a sharp transition in the parameter λ. Namely, for $\lambda < \lambda_{cr}(\alpha)$, there exists a smooth invariant curve (analytic if f is analytic), for $\lambda = \lambda_{cr}(\alpha)$, an invariant curve still exists but it is not C^∞ anymore (perhaps only $C^{1+\epsilon}$-smooth), so that the conjugacy with the rigid rotation is only topological, and the invariant measure is singular. Finally, for $\lambda > \lambda_{cr}(\alpha)$ the minimizers form a Cantor-type set of zero Lebesgue measure. In fact it is even expected that the Hausdorff dimension of such sets vanishes. It is also expected that the dynamical properties of minimizing trajectories are very different before and after the transition. The trajectories belonging to smooth invariant curves are elliptic with zero Lyapunov exponents, while trajectories belonging to cantori are expected to be hyperbolic. Although the above transition is confirmed by many numerical studies, at present there are very few rigorous results in this direction, especially related to the critical case $\lambda = \lambda_{cr}(\alpha)$.

Minimization of the Lagrangian action leads to the discrete Euler-Lagrange equation. In particular, any minimizing sequence $(\varphi_n)_{n \in \mathbb{Z}}$ must be related to a trajectory of the map $\psi_{\lambda f}$. The second differential of the action functional can be written as a quadratic form $(\mathcal{H}u, u)$, where $\mathcal{H}$ is the 1D Schrödinger operator

$$\mathcal{H} \colon (u_n)_{n \in \mathbb{Z}} \mapsto (u_{n+1} + u_{n-1} + V_0(\varphi_n)u_n)_{n \in \mathbb{Z}},$$

with $V_0 := -f' - 2$.

Since a minimizing trajectory is quasi-periodic, the corresponding Schrödinger operator will be an operator with a quasi-periodic potential. In fact, this construction leads to a one-parameter family of such potentials parametrized by the coupling constant λ. It turns out that the potentials are smooth for $\lambda < \lambda_{cr}(\alpha)$ and discontinuous in the case $\lambda > \lambda_{cr}(\alpha)$.

In this paper we propose to study the spectral properties of such families of Schrödinger operators. We prove that in the KAM regime, when there exists an analytic invariant curve, the Schrödinger operator has a component of absolutely continuous spectrum. We construct such a component near the edge of the spectrum. More precisely the following Main Theorem holds.

Main Theorem. Let f be an analytic 1-periodic function with zero mean value. Suppose the standard-type map $\psi_{\lambda f}$ has an analytic invariant curve homotopic to the base with a rotation number α of Brjuno type (see (3.1) for a definition). Then the energy $E = 0$ belongs to the spectrum of the Schrödinger operator $\mathcal{H}$, $E = 0$ is the right edge of the spectrum: $\Sigma(\mathcal{H}) \cap (0, +\infty) = \emptyset$, and there exists $\varepsilon_0(f) > 0$ such that the spectral measures on $[-\varepsilon_0, 0]$ are purely absolutely continuous (and positive).

A more technical formulation of the theorem will be given in Section 3. The proof is based on a dynamical argument. It is easy to see that a Schrödinger cocycle for the energy $E = 0$ is conjugated to the dynamical (Jacobi) cocycle associated to $\psi_{\lambda f}$. This allows us to show that it is, in fact, reducible to a constant parabolic cocycle. Then, using the arguments developed by Avila [4], we can conclude that the spectrum is absolutely continuous in a neighborhood of $E = 0$.

This component of absolutely continuous spectrum comes from almost reducibility properties of Schrödinger cocycles for energies near $E = 0$. In [3], Avila showed that almost reducibility is stable among analytic quasi-periodic $\mathrm{SL}(2, \mathbb{R})$-cocycles with irrational frequency. In a similar vein, the component of absolutely continuous spectrum we obtain is also stable in the following sense: when the invariant curve has Diophantine rotation number α, it persists under small analytic perturbations of the potential. Moreover, by [13], this curve is also accumulated by other analytic invariant curves with Diophantine rotation numbers close to α. Our Main Theorem guarantees stability in both senses, namely, under small analytic perturbations of the potential, and for these Diophantine rotation numbers, the associated Schrödinger operator also has a component of absolutely continuous spectrum.

We view this theorem as a semi-global result. Namely we do not assume that the potential in the Schrödinger operator is small. We only use the existence of an analytic invariant curve. We also conjecture that the critical value $\lambda_{cr}(\alpha)$ is a transition point. In other words, it is plausible that for all $\lambda > \lambda_{cr}(\alpha)$ the spectrum will be pure point. This would mean that the dynamical transition from elliptic to hyperbolic behavior in the weak KAM theory reflects in a related transition in the spectral properties of the corresponding Schrödinger operators. It is an interesting problem to analyze the spectrum at the critical value $\lambda = \lambda_{cr}(\alpha)$.

2. Aubry-Mather theory and Schrödinger operators

2.1. Conservative twist maps of the cylinder.
Let $C^\omega(\mathbb{T}, \mathbb{R})$ be the set of one-periodic analytic functions on $\mathbb{R}$, and let $C_0^\omega(\mathbb{T}, \mathbb{R}) \subset C^\omega(\mathbb{T}, \mathbb{R})$ be the subset of functions with zero average. For any function $f \in C_0^\omega(\mathbb{T}, \mathbb{R})$, we let $\Psi = \Psi_f \colon \mathbb{R}^2 \to \mathbb{R}^2$, $(x, r) \mapsto (x + r + f(x), r + f(x))$. It induces a map on the cylinder $\mathbb{A} := \mathbb{T} \times \mathbb{R}$:

$$\psi = \psi_f \colon \left\{ \begin{array}{rcl} \mathbb{A} & \to & \mathbb{A}, \\ (\varphi, r) & \mapsto & (\varphi + r + f(\varphi) \bmod 1, r + f(\varphi)). \end{array} \right.$$

The map ψ satisfies the twist property: for any $(\varphi, r) \in \mathbb{A}$, we have $\partial_r(\varphi + r + f(\varphi)) = 1 > 0$. For an arbitrary $f \in C_0^\omega(\mathbb{T}, \mathbb{R})$, it is also natural to define a one-parameter family of twist maps $(\psi_{\lambda f})_{\lambda \in \mathbb{R}}$, where λ is called the *coupling constant*.

An important case corresponds to the function $f_0 \colon \varphi \mapsto \frac{1}{2\pi} \sin(2\pi\varphi)$. In this case, for any $\lambda \in \mathbb{R}$, the map $\psi_\lambda := \psi_{\lambda f_0}$ is the *Standard Map* with parameter λ.

Given a point $(x, r) = (x_0, r_0) \in \mathbb{R}^2$, we let $((x_n, r_n))_{n \in \mathbb{Z}}$ be its orbit under Ψ:

$$(2.1) \qquad \left\{ \begin{array}{rcl} x_{n+1} & = & x_n + r_{n+1}, \\ r_{n+1} & = & r_n + f(x_n). \end{array} \right.$$

Similarly, for $(\varphi, r) = (\varphi_0, r_0) \in \mathbb{A}$, we denote by $((\varphi_n, r_n))_{n \in \mathbb{Z}}$ its orbit under ψ; for $x_0 \in \mathbb{R}$ such that $x_0 \bmod 1 = \varphi_0$, we have $(\varphi_n, r_n) = (x_n \bmod 1, r_n)$, for all $n \in \mathbb{Z}$.

Let $f \in C_0^\omega(\mathbb{T}, \mathbb{R})$. The matrix of the differential of $\psi = \psi_f$ at $(\varphi, r) \in \mathbb{A}$ is

$$D\psi_{(\varphi, r)} = \begin{pmatrix} 1 + f'(\varphi) & 1 \\ f'(\varphi) & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),$$

hence ψ is an analytic volume-preserving diffeomorphism with zero Calabi invariant: $\psi \in \mathrm{Diff}_{m,0}^\omega(\mathbb{A})$, where $dm = d\varphi\, dr$ is the Lebesgue measure.

Let $F \in C_0^\omega(\mathbb{T}, \mathbb{R})$ satisfy $F' = f$. For any $a \in \mathbb{R}$, we define a function $h_{f,a} \colon \mathbb{R}^2 \to \mathbb{R}$ by the formula

$$h_{f,a}(x_0, x_1) := \frac{1}{2}(x_1 - x_0 - a)^2 + F(x_0), \qquad \forall \, (x_0, x_1) \in \mathbb{R}^2.$$

Let $(\varphi_0, r_0) \in \mathbb{A}$, and let $(\varphi_1, r_1) := \psi(\varphi_0, r_0)$. Given $x_0 \in \mathbb{R}$ such that $x_0 \bmod 1 = \varphi_0$, we set $(x_1, r_1) := \Psi_f(x_0, r_0)$. The value $h_{f,a}(x_0, x_1)$ is independent of the choice of the lift $x_0 \in \mathbb{R}$ of φ_0, and thus, we may define

$$h_{f,a}(\varphi_0, \varphi_1) := h_{f,a}(x_0, x_1).$$

In particular, given any two consecutive points (φ_n, r_n), (φ_{n+1}, r_{n+1}) in the orbit of $(\varphi, r) = (\varphi_0, r_0) \in \mathbb{A}$, the function $h_{f,a}(\varphi_n, \varphi_{n+1})$ is well-defined.

Moreover, the function $h_{f,a}$ is *generating* for $\psi = \psi_f$ in the following sense: for $(\varphi_0, r_0) \in \mathbb{A}$ and $(\varphi_1, r_1) := \psi(\varphi_0, r_0)$, we have

$$\begin{cases} \partial_1 h_{f,a}(\varphi_0, \varphi_1) &= -(r_0 - a), \\ \partial_2 h_{f,a}(\varphi_0, \varphi_1) &= (r_1 - a). \end{cases}$$

2.2. **Action-minimizing Aubry Mather sets.** Let $f \in C_0^\omega(\mathbb{T}, \mathbb{R})$, and let $F \in C_0^\omega(\mathbb{T}, \mathbb{R})$ be the anti-derivative of f with zero average. Given $a \in \mathbb{R}$, we define the action of a sequence $u = (u_n)_{n \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ as a formal sum:

$$A_{f,a}(u) := \sum_{n \in \mathbb{Z}} h_{f,a}(u_n, u_{n+1}) = \frac{1}{2} \sum_{n \in \mathbb{Z}} (u_{n+1} - u_n - a)^2 + \sum_{n \in \mathbb{Z}} F(u_n).$$

The sequence $u = (u_n)_{n \in \mathbb{Z}}$ is called a *minimizer of the action $A_{f,a}$* if for any compact perturbation $\tilde{u} = (\tilde{u}_n)_{n \in \mathbb{Z}}$ of $(u_n)_{n \in \mathbb{Z}}$, the difference in action satisfies $A_{f,a}(\tilde{u}) - A_{f,a}(u) \geq 0$. Notice that the difference of actions is well defined although $A_{f,a}$ itself is just a formal series.

Recall that minimizers of the action are associated to orbits of Ψ_f:

Lemma 2.1. *Let $a \in \mathbb{R}$, and assume that the sequence $(x_n)_{n \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ is a minimizer of the action $A_{f,a}$. We set $r_n := x_n - x_{n-1}$, for all $n \in \mathbb{Z}$. Then, $((x_n, r_n))_{n \in \mathbb{Z}}$ is the orbit of (x_0, r_0) under Ψ_f, and its projection $((\varphi_n, r_n))_{n \in \mathbb{Z}}$ on $\mathbb{A}$ is the orbit of (φ_0, r_0) under ψ_f, with $\varphi_n := x_n \bmod 1$, for all $n \in \mathbb{Z}$.*

Proof. Given any $u = (u_n)_{n \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $\delta = (\delta_n)_{n \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ satisfying $\delta_n = 0$ for all but finitely many integers $n \in \mathbb{Z}$, with $\|\delta\| := \sqrt{\sum_k \delta_k^2} \ll 1$, we obtain

$$(2.2) \quad A_{f,a}(u + \delta) - A_{f,a}(u) = -\sum_{n \in \mathbb{Z}} (u_{n+1} - 2u_n + u_{n-1} - f(u_n))\delta_n + O(\|\delta\|^2),$$

where $(u + \delta)_n := u_n + \delta_n$. Now, assume that the sequence $(x_n)_{n \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ is a minimizer of the action $A_{f,a}$, and set $r_n := x_n - x_{n-1}$, for all $n \in \mathbb{Z}$. Since δ can be taken arbitrarily small, we deduce that

$$r_{n+1} = x_{n+1} - x_n = x_n - x_{n-1} + f(x_n) = r_n + f(x_n),$$
$$x_{n+1} = 2x_n - x_{n-1} + f(x_n) = x_n + r_{n+1},$$

for each $n \in \mathbb{Z}$. As a result, (2.1) is satisfied, and $((x_n, r_n))_{n \in \mathbb{Z}}$ is the orbit of (x_0, r_0) under Ψ_f. Then, $((\varphi_n, r_n))_{n \in \mathbb{Z}}$ is the orbit of (φ_0, r_0) under ψ_f, with $\varphi_n := x_n \bmod 1$, for all $n \in \mathbb{Z}$. $\qquad \square$

Remark 2.2. *The calculation above is just a derivation of the discrete Euler-Lagrange equation associated with the action given by $A_{f,a}$.*

A ψ_f-invariant compact set $\mathcal{A} \subset \mathbb{A}$ is said to be ψ_f-*ordered* if it projects injectively on $\mathbb{T}$, and the restriction $\psi_f|_{\mathcal{A}}$ preserves the natural order given by the projection. A classical result of Aubry-Mather theory [1, 2, 25] states that for each irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exist $a = a(\alpha) \in \mathbb{R}$ and a minimal ordered set $\mathcal{AM}_{f,\alpha} \subset \mathbb{A}$ with rotation number α. It is comprised of orbits $((\varphi_n, r_n))_{n \in \mathbb{Z}}$ under ψ_f associated to sequences $(x_n)_{n \in \mathbb{Z}}$ which minimize the action $A_{f,a}$. By some slight abuse of notation, we denote $A_{f,\alpha} := A_{f,a(\alpha)}$ in the following. In particular, the rotation number α of $\mathcal{AM}_{f,\alpha}$ is the rotation number of any lifted orbit: for any $(\varphi_0, r_0) = (x_0 \bmod 1, r_0) \in \mathcal{AM}_{f,\alpha}$, we have

$$\alpha = \lim_{n \to +\infty} \frac{x_n - x_0}{n} = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} r_k.$$

The set $\mathcal{AM}_{f,\alpha}$ is called the *minimizing Aubry-Mather set* for the action $A_{f,\alpha}$.

Theorem 2.3 ([1, 2, 9, 16, 19, 25]). *For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the associated minimizing Aubry-Mather set $\mathcal{AM}_{f,\alpha}$ is either an invariant graph Γ_γ for some Lipschitz function $\gamma \colon \mathbb{T} \to \mathbb{R}$, or it projects one-to-one to a nowhere-dense Cantor set of $\mathbb{T}$. Moreover, if Γ is an invariant curve for ψ_f homotopic to the base with irrational rotation number α, it is a minimizing Aubry-Mather set: we have $\Gamma = \mathcal{AM}_{f,\alpha}$.*

Suppose that ψ_f leaves invariant the graph $\Gamma_\gamma = \{\Gamma_\gamma(\varphi) := (\varphi, \gamma(\varphi)) : \varphi \in \mathbb{T}\}$[1] with rotation number $\alpha \bmod 1$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then, the composition $\pi_1 \circ \psi_f \circ \Gamma_\gamma$ yields a circle homeomorphism $g = g_\gamma \colon \mathbb{T} \to \mathbb{T}$, where $\pi_1 \colon \mathbb{A} \to \mathbb{T}$ is the projection on the first coordinate:

$$g(\varphi) := \varphi + \gamma(\varphi) + f(\varphi) \bmod 1, \qquad \forall \varphi \in \mathbb{T}.$$

The map $G = G_\gamma \colon \mathbb{R} \ni x \mapsto x + \gamma(x) + f(x)$ is a lift of g, and it has rotation number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Since g has rotation number $\alpha \bmod 1$, it is uniquely ergodic. We denote by ν its unique invariant probability measure. For any orbit $((\varphi_n, r_n))_{n \in \mathbb{Z}} \subset \mathcal{AM}_{f,\alpha}$, we have $r_k = \gamma(\varphi_k) = \gamma(g^k(\varphi_0))$, for all $k \in \mathbb{Z}$, hence

$$\alpha = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} r_k = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \gamma(g^k(\varphi_0)) = \int_{\mathbb{T}} \gamma(\varphi) \, d\nu(\varphi).$$

2.3. **Dynamically defined quasi-periodic Schrödinger operators.** Let $f \in C_0^\omega(\mathbb{T}, \mathbb{R})$. Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let us consider the minimizing Aubry-Mather set $\mathcal{AM}_{f,\alpha}$. In the following, we choose a phase $\varphi_0 \in \mathbb{T}$ such that $(\varphi_0, r_0) \in \mathcal{AM}_{f,\alpha}$ for some $r_0 \in \mathbb{R}$. Given $x_0 \in \mathbb{R}$ such that $\varphi_0 = x_0 \bmod 1$, we denote by $((x_n, r_n))_{n \in \mathbb{Z}}$ the orbit of (x_0, r_0) under Ψ_f and let $((\varphi_n, r_n))_{n \in \mathbb{Z}}$ be the corresponding ψ_f-orbit. We also denote by $(\cdot, \cdot)$ the standard inner product on $\ell^2(\mathbb{Z})$, and for $u \in \ell^2(\mathbb{Z})$, we set $\|u\| := \sqrt{(u, u)}$.

In the proof of Lemma 2.1, we have computed the first order term in the difference of actions between a sequence and a compact perturbation of it. It turns out that

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[1]We shall use the same notation Γ_γ for the natural map from $\mathbb{T}$ to $\mathbb{A}$ defined by the graph Γ_γ.

the second order term in this difference is given by some quadratic form associated to a Schrödinger operator, as shown by the next lemma.

Lemma 2.4. *For any sequence $\delta = (\delta_n)_{n \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ satisfying $\delta_n = 0$ for all but finitely many integers $n \in \mathbb{Z}$ and such that $\|\delta\| \ll 1$, we have*

$$(2.3) \qquad A_{f,\alpha}((x_n + \delta_n)_{n \in \mathbb{Z}}) - A_{f,\alpha}((x_n)_{n \in \mathbb{Z}}) = -\frac{1}{2}(\mathcal{H}_{f,\alpha,\varphi_0}\delta, \delta) + O(\|\delta\|^3),$$

where $\mathcal{H}_{f,\alpha,\varphi_0}$ is the dynamically defined Schrödinger operator associated to the analytic function $V_0 := -f' - 2$ and the ψ_f-ordered sequence $(\varphi_n)_{n \in \mathbb{Z}}$:

$$\mathcal{H}_{f,\alpha,\varphi_0} \colon \begin{cases} \ell^2(\mathbb{Z}) & \to & \ell^2(\mathbb{Z}), \\ (u_n)_{n \in \mathbb{Z}} & \mapsto & (u_{n+1} + u_{n-1} + V_0(\varphi_n)u_n)_{n \in \mathbb{Z}}. \end{cases}$$

Proof. Since $((x_n, r_n))_{n \in \mathbb{Z}}$ is a Ψ_f-orbit, we have $x_{n+1} - 2x_n + x_{n-1} - f(x_n) = 0$, for all $n \in \mathbb{Z}$. Equivalently, $(x_n)_{n \in \mathbb{Z}}$ is a critical point for the action $A_{f,\alpha}$, and the first order term in the difference (2.2) vanishes for $(x_n)_{n \in \mathbb{Z}}$ in place of $(u_n)_{n \in \mathbb{Z}}$. Given any compact perturbation $(x_n + \delta_n)_{n \in \mathbb{Z}}$ of $(x_n)_{n \in \mathbb{Z}}$ for some sequence $\delta = (\delta_n)_{n \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ with $\|\delta\| \ll 1$, the Taylor expansion of the difference in action is

$$A_{f,\alpha}((x_n + \delta_n)_{n \in \mathbb{Z}}) - A_{f,\alpha}((x_n)_{n \in \mathbb{Z}}) = \sum_{n \in \mathbb{Z}} \frac{1}{2}(\delta_{n+1} - \delta_n)^2 + f'(x_n)\frac{\delta_n^2}{2} + O(\|\delta\|^3)$$

$$= -\frac{1}{2}(\mathcal{H}_{f,\alpha,\varphi_0}\delta, \delta) + O(\|\delta\|^3).$$

$\square$

In particular, for any ψ_f-ordered sequence $(\varphi_n)_{n \in \mathbb{Z}} \subset \mathcal{AM}_{f,\alpha}$, $\mathcal{H} = \mathcal{H}_{f,\alpha,\varphi_0}$ is a quasi-periodic Schrödinger operator whose potential is related to the dynamics of $\psi_f|_{\mathcal{AM}_{f,\alpha}}$. We denote by $\Sigma = \Sigma(\mathcal{H})$ its *spectrum*; recall that it is the set of energies E such that the operator $\mathcal{H} - E = \mathcal{H} - E \cdot I$ does not have a bounded inverse in $\ell^2(\mathbb{Z})$. For any $u \in \ell^2(\mathbb{Z})$, we denote by $\mu_{\mathcal{H}}^u$ the spectral measure of $\mathcal{H}$ associated to u. It is defined by the following formula:

$$((\mathcal{H} - E)^{-1}u, u) = \int_{\mathbb{R}} \frac{1}{E' - E} d\mu_{\mathcal{H}}^u(E'),$$

for any energy E in the resolvent set $\mathbb{C} \setminus \Sigma$. The union of the supports of all spectral measures is equal to Σ. We refer for instance to [10] for more details on dynamically defined Schrödinger operators.

Lemma 2.5. *For any $u \in \ell^2(\mathbb{Z})$, we have*

$$(2.4) \qquad\qquad\qquad\qquad (\mathcal{H}u, u) \leq 0.$$

Equivalently,

$$(2.5) \qquad\qquad\qquad\qquad \Sigma(\mathcal{H}) \subset (-\infty, 0].$$

Proof. We first show (2.4). Let $u \in \ell^2(\mathbb{Z})$. For any integer $n_0 \geq 1$, we define the compact sequence $u^{n_0} = (u_n^{n_0})_{n \in \mathbb{Z}}$, where $u_n^{n_0} := u_n$ if $n \in [-n_0, n_0]$, and $u_n^{n_0} := 0$ otherwise. As $(x_n)_{n \in \mathbb{Z}}$ is a minimizer of $A_{f,\alpha}$, for any integer $k \geq 1$, the following expression is always nonnegative:

$$A_{f,\alpha}\left(\left(x_n + \frac{1}{k}u_n^{n_0}\right)_{n \in \mathbb{Z}}\right) - A_{f,\alpha}((x_n)_{n \in \mathbb{Z}}) = -\frac{1}{2k^2}(\mathcal{H}u^{n_0}, u^{n_0}) + O\left(\frac{1}{k^3}\|u^{n_0}\|^3\right).$$

For $k \gg 1$ large, and as the error term in the above Taylor expansion scales like $\frac{1}{k^3}$, we deduce that $(\mathcal{H}u^{n_0}, u^{n_0}) \leq 0$. Letting $n_0 \to +\infty$, we conclude that $(\mathcal{H}u, u) \leq 0$.

We now prove the second statement (2.5) about the spectrum $\Sigma = \Sigma(\mathcal{H})$. It amounts to showing that for any $E > 0$, the operator $\mathcal{H} - E$ is invertible and that its inverse is bounded. Fix $E > 0$. By (2.4), for any sequence $u \in \ell^2(\mathbb{Z})$, we have

$$(2.6) \qquad \|(\mathcal{H} - E)u\|^2 = \|\mathcal{H}u\|^2 - 2E(\mathcal{H}u, u) + E^2\|u\|^2 \geq E^2\|u\|^2.$$

Therefore, if $(\mathcal{H} - E)u = 0$, then $\|u\| = 0$, and thus, $\mathcal{H} - E$ is injective. Moreover, (2.6) also implies that the inverse of $\mathcal{H} - E$ is bounded, since

$$\sup_{u \in \ell^2(\mathbb{Z})} \frac{\|(\mathcal{H} - E)^{-1}u\|}{\|u\|} = \left(\inf_{u \in \ell^2(\mathbb{Z})} \frac{\|(\mathcal{H} - E)u\|}{\|u\|} \right)^{-1} \leq \frac{1}{E}.$$

Besides, $\mathcal{H} - E$ is automatically surjective. Indeed, it holds $\mathrm{Im}(\mathcal{H} - E)^\perp = \mathrm{Ker}(\mathcal{H} - E)^* = \mathrm{Ker}(\mathcal{H} - E) = \{0\}$, hence $\overline{\mathrm{Im}(\mathcal{H} - E)} = \ell^2(\mathbb{Z})$. By (2.6) and Cauchy's convergence test, we deduce that $\mathrm{Im}(\mathcal{H} - E) = \ell^2(\mathbb{Z})$. We conclude that $\Sigma \subset (-\infty, 0]$.

Conversely, let us show that $\Sigma \subset (-\infty, 0]$ implies (2.4). Indeed, the spectrum Σ is the union of the supports of the spectral measures $\{\mu_\mathcal{H}^u\}_{u \in \ell^2(\mathbb{Z})}$, hence for any $u \in \ell^2(\mathbb{Z})$, the support of $\mu_\mathcal{H}^u$ is contained in $(-\infty, 0]$, and

$$(\mathcal{H}u, u) = \int_\mathbb{R} E \, d\mu_\mathcal{H}^u(E) = \int_{-\infty}^0 E \, d\mu_\mathcal{H}^u(E) \leq 0.$$

$\square$

3. Main Theorem

Let $f \in C_0^\omega(\mathbb{T}, \mathbb{R})$, and let $\psi_f \colon \mathbb{A} \to \mathbb{A}$ be the associated twist map. Assume that ψ_f leaves invariant an analytic curve Γ which is the graph $\Gamma = \Gamma_\gamma := \{(\varphi, \gamma(\varphi)) : \varphi \in \mathbb{T}\}$ of some function $\gamma \in C^\omega(\mathbb{T}, \mathbb{R})$. We denote by

$$g \colon \varphi \mapsto \varphi + \gamma(\varphi) + f(\varphi) \bmod 1$$

the analytic diffeomorphism of $\mathbb{T}$ induced by $\psi_f|_{\Gamma_\gamma}$, and assume that the rotation number α of g satisfies the *Brjuno condition*, i.e.,

$$(3.1) \qquad \mathfrak{B}(\alpha) := \sum_{k=0}^{+\infty} \frac{1}{q_k} \log q_{k+1} < +\infty,$$

where $\left(\frac{p_k}{q_k}\right)_{k \geq 0}$ denotes the sequence of convergents for α (corresponding to the continued fraction algorithm). In particular, we can use the results of [3, 7, 24], where it is assumed that $\beta(\alpha) := \limsup_k \frac{1}{q_k} \log q_{k+1} = 0$.

As above, given $\varphi_0 \in \mathbb{T}$, we consider the Schrödinger operator

$$\mathcal{H} \colon (u_n)_{n \in \mathbb{Z}} \mapsto (u_{n+1} + u_{n-1} + V_0(g^n(\varphi_0))u_n)_{n \in \mathbb{Z}},$$

with $V_0 := -f' - 2$, and denote by $\Sigma(\mathcal{H})$ its spectrum.

Our main result is:

Main Theorem. *Assume that the map ψ_f leaves invariant an analytic curve Γ with Brjuno rotation number α, and let $\mathcal{H}$ be the associated Schrödinger operator. Then there exists $\varepsilon_0 > 0$ such that the following properties hold:*

    (1) *the energy $E = 0$ is the right edge of the spectrum: $0 = \max \Sigma(\mathcal{H})$;*

    (2) *the spectral measures of $\mathcal{H}$ restricted to $[-\varepsilon_0, 0]$ are absolutely continuous;*

    (3) *there exists $\kappa > 0$ such that $|(E - \varepsilon, E + \varepsilon) \cap \Sigma(\mathcal{H})| > \kappa\varepsilon$, for all energy $E \in \Sigma(\mathcal{H}) \cap (-\varepsilon_0, 0)$, and for all $0 < \varepsilon < |E|$.*

The proof is based on the reducibility of the *Schrödinger cocycle* associated to $\mathcal{H}$ for the energy $E = 0$. As we shall explain, this can be seen in two ways:

    (1) in restriction to Γ, the *Jacobi (differential) cocycle* of the twist map ψ_f is conjugate to some quasi-periodic Schrödinger cocycle (see Subsections 4.1-4.2). Besides, in Subsection 4.3, we use a vector field tangent to Γ to reduce these cocycles to a constant cocycle associated to some parabolic matrix.

    (2) the energy $E = 0$ is in the pure point spectrum of some *dual Schrödinger operator* (see Subsection 4.4). Actually, thanks to the existence of the invariant curve Γ, we construct an explicit eigenvector whose coefficients decay exponentially fast.

By Avila's results, this implies that for small energies, the corresponding Schrödinger cocycles are *almost reducible*, i.e., they can be conjugated uniformly in some strip to a cocycle which is arbitrarly close to a constant. Finally, we follow the proof given by Avila in [4] to show the existence of a component of absolutely continuous spectrum near the energy $E = 0$.

Remark 3.1. • *Such properties are typical of the regime of small analytic potentials (see [4, 7, 12] for instance). Our result replaces the usual smallness assumption with the geometric assumption on the existence of an analytic invariant curve.*

    • *Let us also recall that for maps ψ_f as above, the existence of analytic invariant curves with a given Brjuno rotation number is guaranteed by the main result of [17], provided that the analytic norm of f is sufficiently small. More precisely, by Theorem 1.1 in [17], for any $f_0 \in C_0^\omega(\mathbb{T}, \mathbb{R})$ and for any $\alpha \in \mathbb{R}$ satisfying the Brjuno condition $\mathfrak{B}(\alpha) < +\infty$, there exists $\lambda_0 > 0$ such that for $|\lambda| < \lambda_0$, the map $\psi_{\lambda f_0}$ admits an analytic invariant curve with rotation number α.*

4. Invariant curves & almost reducibility of Schrödinger cocycles

4.1. Invariant curves & Jacobi (differential) cocycle. Let $f \in C_0^\omega(\mathbb{T}, \mathbb{R})$ be an analytic function with zero average. For the map $\psi = \psi_f$ one can define in a usual way the Jacobi cocycle $(\psi, D\psi) \colon \mathbb{A} \times \mathbb{C}^2 \to \mathbb{A} \times \mathbb{C}^2$, namely

$$(\psi, D\psi)((\varphi, r), v) := (\psi(\varphi, r), D\psi_{(\varphi, r)} \cdot v), \quad \forall ((\varphi, r), v) \in \mathbb{A} \times \mathbb{C}^2.$$

Assume that $\Gamma \subset \mathbb{A}$ is an invariant curve for ψ homotopic to the base. As recalled in Theorem 2.3, by Birkhoff Theorem, Γ is the graph $\Gamma_\gamma = \{\Gamma_\gamma(\varphi) = (\varphi, \gamma(\varphi)) : \varphi \in \mathbb{T}\}$ of some Lipschitz function $\gamma \colon \mathbb{T} \to \mathbb{R}$. As above, we let $g = g_\gamma \colon \varphi \mapsto \varphi + \gamma(\varphi) + f(\varphi) \mod 1$ be the circle map obtained by projecting $\psi|_{\Gamma_\gamma}$ on the first coordinate. For any $\varphi \in \mathbb{T}$, we have

$$(4.1) \qquad \psi(\Gamma_\gamma(\varphi)) = \psi(\varphi, \gamma(\varphi)) = (g(\varphi), \gamma(\varphi) + f(\varphi)) = (g(\varphi), \gamma \circ g(\varphi)) \in \Gamma_\gamma.$$

Clearly, g is a homeomorphism of $\mathbb{T}$; moreover, g and γ have the same regularity. In particular, in the case we consider, γ and g are analytic.

As the curve Γ is invariant under ψ_f, the restriction of the Jacobi cocycle to Γ reduces to the derivative cocycle $(g, D\psi) \colon \mathbb{T} \times \mathbb{C}^2 \to \mathbb{T} \times \mathbb{C}^2$:

$$(4.2) \qquad (g, D\psi)(\varphi, v) := (g(\varphi), D\psi(\varphi) \cdot v), \quad \forall\, (\varphi, v) \in \mathbb{T} \times \mathbb{C}^2,$$

where $D\psi(\varphi) := D\psi_{(\varphi, \gamma(\varphi))}$. Let us denote by v_0 a vector field tangent to the invariant curve:

$$v_0 \colon \varphi \mapsto \begin{pmatrix} 1 \\ \gamma'(\varphi) \end{pmatrix} \in T_{(\varphi, \gamma(\varphi))}\Gamma.$$

It is easy to see that v_0 is an invariant section for the derivative cocycle in the directional sense.

Lemma 4.1. *The action of the cocycle $(g, D\psi)$ on the vector field v_0 is given by*

$$(4.3) \qquad D\psi(\varphi) \cdot v_0(\varphi) = g'(\varphi) \cdot v_0(g(\varphi)), \quad \forall\, \varphi \in \mathbb{T},$$

with $g'(\varphi) = 1 + \gamma'(\varphi) + f'(\varphi) = (1 - \gamma' \circ g(\varphi))^{-1}$.

Moreover, there exists an analytic conjugacy map $Z_1 \in C^\omega(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$ such that

$$(4.4) \qquad (Z_1 \circ g(\varphi))^{-1} D\psi(\varphi) Z_1(\varphi) = \begin{pmatrix} g'(\varphi) & 1 \\ 0 & g'(\varphi)^{-1} \end{pmatrix}, \quad \forall\, \varphi \in \mathbb{T}.$$

Proof. As we have seen in (4.1), the fact that Γ is invariant means that f can be written as a coboundary, i.e., it satisfies the cohomological equation

$$(4.5) \qquad f(\varphi) = \gamma \circ g(\varphi) - \gamma(\varphi), \quad \forall\, \varphi \in \mathbb{T}.$$

Differentiating with respect to φ in (4.5), we obtain:

$$(4.6) \qquad \gamma' \circ g(\varphi) \cdot g'(\varphi) = \gamma'(\varphi) + f'(\varphi),$$

with $g'(\varphi) = 1 + \gamma'(\varphi) + f'(\varphi)$. It also follows that $g'(\varphi) \cdot (1 - \gamma' \circ g(\varphi)) = 1$.

By (4.6), we deduce that

$$\begin{pmatrix} 1 + f'(\varphi) & 1 \\ f'(\varphi) & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \gamma'(\varphi) \end{pmatrix} = g'(\varphi) \begin{pmatrix} 1 \\ \gamma' \circ g(\varphi) \end{pmatrix},$$

i.e., $D\psi(\varphi) \cdot v_0(\varphi) = g'(\varphi) \cdot v_0(g(\varphi))$.

Now, let us set

$$Z_1(\varphi) := \begin{pmatrix} 1 & 0 \\ \gamma'(\varphi) & 1 \end{pmatrix}, \quad \forall\, \varphi \in \mathbb{T}.$$

Then, by (4.3), and since $(g'(\varphi))^{-1} = 1 - \gamma' \circ g(\varphi)$, we see that the map Z_1 conjugates $(g, D\psi)$ to the upper-triangular cocycle as in (4.4). $\qquad\square$

4.2. Schrödinger cocycle associated to an invariant curve. Let f, $\psi = \psi_f$, Γ and g be as in the previous subsection. As we have seen above, one can define a natural family of dynamically generated Schrödinger operators, the phase $\varphi_0 \in \mathbb{T}$ being a parameter:

$$\mathcal{H}_{f, \alpha, \varphi_0} \colon (u_n)_{n \in \mathbb{Z}} \mapsto (u_{n+1} + u_{n-1} + V_0(g^n(\varphi_0))u_n)_{n \in \mathbb{Z}},$$

where $V_0 := -f' - 2$. For an arbitrary energy $E \in \mathbb{R}$, one can define a Schrödinger cocycle $(g, S_E^{V_0})$, with

$$S_E^{V_0}(\varphi) := \begin{pmatrix} E - V_0(\varphi) & -1 \\ 1 & 0 \end{pmatrix}, \quad \forall \varphi \in \mathbb{T}.$$

It acts on $\mathbb{T} \times \mathbb{C}^2$ by the following formula:

$$(g, S_E^{V_0})(\varphi, v) := (g(\varphi), S_E^{V_0}(\varphi) \cdot v), \quad \forall (\varphi, v) \in \mathbb{T} \times \mathbb{C}^2.$$

It turns out that the Schrödinger cocycle and the derivative cocycle are conjugate to each other for the energy $E = 0$: for any $\varphi \in \mathbb{T}$, we have

(4.7) $$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 + f'(\varphi) & 1 \\ f'(\varphi) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 + f'(\varphi) & -1 \\ 1 & 0 \end{pmatrix}.$$

We next show that both cocycles $(g, S_0^{V_0})$ and $(g, D\psi)$ are conjugate to a constant parabolic cocycle (g, B_0), for some matrix $B_0 := \begin{pmatrix} 1 & \nu_0 \\ 0 & 1 \end{pmatrix}$. This conjugation holds only along the invariant curve. To proceed, we need some facts from the Herman-Yoccoz theory.

Assume that the rotation number α of Γ satisfies the Brjuno condition $\mathfrak{B}(\alpha) := \sum_{k=0}^{+\infty} \frac{1}{q_k} \log q_{k+1} < +\infty$, where $\left(\frac{p_k}{q_k}\right)_{k \geq 0}$ denotes the sequence of continued fraction convergents for α. We shall also assume that $\Gamma = \Gamma_\gamma$ is the graph of some analytic function $\gamma \in C^\omega(\mathbb{T}, \mathbb{R})$. It follows that the circle diffeomorphism $g = g_\gamma$ is also analytic. Hence, by the theorem of Yoccoz (see [28] and also [14,15] for a reference), g is analytically conjugate to the rigid rotation r_α by angle α. Namely, there exists an analytic circle diffeomorphism $\phi = \phi_\gamma \in C^\omega(\mathbb{T}, \mathbb{T})$ such that

(4.8) $$\phi^{-1} \circ g \circ \phi(\varphi) = \varphi + \alpha \mod 1, \quad \forall \varphi \in \mathbb{T}.$$

The following lemma says that both f and γ can be expressed in terms of the conjugacy map ϕ.

Lemma 4.2. *It holds*

(4.9) $$f = \gamma \circ g - \gamma = \gamma \circ \phi r_\alpha \phi^{-1} - \gamma.$$

Moreover, we have

(4.10) $$\gamma = \mathrm{I} - g^{-1} = \mathrm{I} - \phi r_\alpha^{-1} \phi^{-1},$$

where I *denotes the identity map, which implies that*

(4.11) $$f = g - 2\mathrm{I} + g^{-1} = \phi r_\alpha \phi^{-1} - 2\mathrm{I} + \phi r_\alpha^{-1} \phi^{-1}.$$

Proof. The identity in (4.9) follows directly from (4.5). As $g = g_\gamma$ is the circle diffeomorphism induced by $\psi_f|_{\Gamma_\gamma}$, for all $\varphi \in \mathbb{T}$, we have $g(\varphi) = \varphi + \gamma(g(\varphi))$, which implies $\gamma = \mathrm{I} - g^{-1}$, and hence, (4.10). Finally, (4.11) follows from (4.9) and (4.10). $\square$

Conversely, given an irrational frequency $\tilde{\alpha}$ and an analytic circle diffeomorphism $\tilde{\phi}$, one can produce an analytic function $\tilde{f}$ with zero average such that the associated twist map $\psi_{\tilde{f}}$ has an analytic invariant curve with rotation number $\tilde{\alpha}$. Moreover, $\tilde{\phi}$ conjugates the corresponding circle diffeomorphism $\tilde{g}$ to the rigid rotation $r_{\tilde{\alpha}}$.

Lemma 4.3. *Let $\tilde{\alpha} \in \mathbb{R} \setminus \mathbb{Q}$ and let $\tilde{\phi} \in C^\omega(\mathbb{T}, \mathbb{T})$ be an analytic circle diffeomorphism. We define two functions $\tilde{\gamma} = \tilde{\gamma}_{\tilde{\alpha}, \tilde{\phi}} \in C^\omega(\mathbb{T}, \mathbb{R})$ and $\tilde{f} = \tilde{f}_{\tilde{\alpha}, \tilde{\phi}} \in C_0^\omega(\mathbb{T}, \mathbb{R})$:*

$$\tilde{\gamma} := \mathrm{I} - \tilde{\phi} r_{\tilde{\alpha}}^{-1} \tilde{\phi}^{-1},$$
$$\tilde{f} := \tilde{\gamma} \circ \tilde{\phi} r_{\tilde{\alpha}} \tilde{\phi}^{-1} - \tilde{\gamma} = \tilde{\phi} r_{\tilde{\alpha}} \tilde{\phi}^{-1} - 2\mathrm{I} + \tilde{\phi} r_{\tilde{\alpha}}^{-1} \tilde{\phi}^{-1}.$$

Then, the analytic graph $\Gamma_{\tilde{\gamma}} := \{\Gamma_{\tilde{\gamma}}(\varphi) = (\varphi, \tilde{\gamma}(\varphi)) : \varphi \in \mathbb{T}\}$ is invariant under $\psi_{\tilde{f}}$ with a rotation number $\tilde{\alpha}$, and $\tilde{\phi}^{-1} \circ \tilde{g} \circ \tilde{\phi} = r_{\tilde{\alpha}}$.

Let $f, \psi, \Gamma = \Gamma_\gamma, g = g_\gamma$ and $\phi = \phi_\gamma$ be as previously. For any map $A \in C^\omega(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$, we denote by (α, A) the associated cocycle over the rigid rotation by angle α. It acts on $\mathbb{T} \times \mathbb{C}^2$ as follows:

$$(\alpha, A)(\varphi, v) := (\varphi + \alpha, A(\varphi) \cdot v), \quad \forall (\varphi, v) \in \mathbb{T} \times \mathbb{C}^2.$$

After conjugation by ϕ, the derivative cocycle and the Schrödinger cocycle yield cocycles $(\alpha, D\psi \circ \phi)$ and (α, S_E^V), $E \in \mathbb{R}$, where $V := V_0 \circ \phi = -f' \circ \phi - 2 \in C^\omega(\mathbb{T}, \mathbb{R})$. Besides, according to (4.7), these cocycles are conjugated by the matrix $M := \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = M^{-1} \in \mathrm{GL}(2, \mathbb{R})$ for the energy $E = 0$:

$$(4.12) \qquad M^{-1} \cdot D\psi(\phi(\varphi)) \cdot M = S_0^V(\varphi), \quad \forall \varphi \in \mathbb{T}.$$

The Schrödinger cocycles $\{(\alpha, S_E^V)\}_{E \in \mathbb{R}}$ are associated to the family of Schrödinger operators $\{H_{V,\alpha,\varphi_0}\}_{\varphi_0 \in \mathbb{T}}$ over the dynamics of the rigid rotation r_α:

$$H_{V,\alpha,\varphi_0} : (u_n)_{n \in \mathbb{Z}} \mapsto (u_{n+1} + u_{n-1} + V(\varphi_0 + n\alpha) u_n)_{n \in \mathbb{Z}}.$$

The spectrum $\Sigma(H_{V,\alpha,\varphi_0})$ does not depend on the phase φ_0. Recall that $\Gamma = \mathcal{AM}_{f,\alpha}$, and note that $H_{V,\alpha,\phi^{-1}(\varphi_0)} = \mathcal{H}_{f,\alpha,\varphi_0}$. In particular, $\Sigma(H_{V,\alpha,\varphi_0}) = \Sigma(\mathcal{H}_{f,\alpha,\varphi_0})$.

4.3. **Parabolic reducibility of the Schrödinger cocycle.** We keep the notations of the previous subsection. As a consequence of Lemma 4.1, we obtain:

Corollary 4.4. *Let $Z_2 := M \cdot Z_1 \circ \phi \in C^\omega(\mathbb{T}, \mathrm{GL}(2, \mathbb{R}))$. Then, we have*

$$(4.13) \qquad Z_2(\varphi + \alpha)^{-1} S_0^V(\varphi) Z_2(\varphi) = \begin{pmatrix} \kappa(\varphi) & 1 \\ 0 & \kappa(\varphi)^{-1} \end{pmatrix}, \quad \forall \varphi \in \mathbb{T},$$

with $\kappa(\varphi) := \frac{\phi'(\varphi + \alpha)}{\phi'(\varphi)}$.

Proof. Recall that $\phi = \phi_\gamma$ satisfies $g \circ \phi(\varphi) = \phi(\varphi + \alpha)$, for all $\varphi \in \mathbb{T}$. Therefore, $g' \circ \phi(\varphi) = \frac{\phi'(\varphi + \alpha)}{\phi'(\varphi)} =: \kappa(\varphi)$. Set $\widetilde{Z}_1 := Z_1 \circ \phi$. By (4.4), we deduce that

$$\widetilde{Z}_1(\varphi + \alpha)^{-1} D\psi(\phi(\varphi)) \widetilde{Z}_1(\varphi) = \begin{pmatrix} \kappa(\varphi) & 1 \\ 0 & \kappa(\varphi)^{-1} \end{pmatrix}, \quad \forall \varphi \in \mathbb{T}.$$

Set $Z_2 := M \cdot \widetilde{Z}_1 = M \cdot Z_1 \circ \phi$. By (4.12), we thus conclude that

$$Z_2(\varphi + \alpha)^{-1} S_0^V(\varphi) Z_2(\varphi) = \begin{pmatrix} \kappa(\varphi) & 1 \\ 0 & \kappa(\varphi)^{-1} \end{pmatrix}, \quad \forall \varphi \in \mathbb{T}.$$

$\square$

As a consequence of the above result, we show that for the energy $E = 0$, the Schrödinger cocycle (α, S_0^V) can be reduced to a parabolic cocycle.

Proposition 4.5. *There exist a negative number $\nu_0 < 0$ and an analytic conjugacy $Z \in C^\omega(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$ homotopic to the identity such that*

$$Z(\varphi + \alpha)^{-1} S_0^V(\varphi) Z(\varphi) = B_0 := \begin{pmatrix} 1 & \nu_0 \\ 0 & 1 \end{pmatrix}, \quad \forall \varphi \in \mathbb{T}.$$

Proof. For any $\varphi \in \mathbb{T}$, and for $\kappa(\varphi) = \frac{\phi'(\varphi + \alpha)}{\phi'(\varphi)}$ as in (4.13), we see that

$$\begin{pmatrix} \phi'(\varphi + \alpha) & 0 \\ 0 & -\phi'(\varphi + \alpha)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \nu(\varphi) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi'(\varphi)^{-1} & 0 \\ 0 & -\phi'(\varphi) \end{pmatrix} = \begin{pmatrix} \kappa(\varphi) & 1 \\ 0 & \kappa(\varphi)^{-1} \end{pmatrix},$$

with $\nu(\varphi) := -(\phi'(\varphi)\phi'(\varphi + \alpha))^{-1} < 0$.

For all $\varphi \in \mathbb{T}$, we let $Z_3(\varphi) := Z_2(\varphi) \cdot \mathrm{diag}(\phi'(\varphi), -(\phi')^{-1}(\varphi))$, such that $Z_3 \in C^\omega(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$. By (4.13), we thus get

$$Z_3(\varphi + \alpha)^{-1} S_0^V(\varphi) Z_3(\varphi) = \begin{pmatrix} 1 & \nu(\varphi) \\ 0 & 1 \end{pmatrix}.$$

Set $\nu_0 := \int_{\mathbb{T}} \nu(\varphi)\, d\varphi < 0$, so that $\nu - \nu_0 \in C_0^\omega(\mathbb{T}, \mathbb{R})$. Since $\mathfrak{B}(\alpha) < +\infty$ and ν is analytic, the following cohomological equation has a solution $\mu \in C^\omega(\mathbb{T}, \mathbb{R})$:

$$\mu(\varphi + \alpha) - \mu(\varphi) = \nu(\varphi) - \nu_0, \quad \forall \varphi \in \mathbb{T}.$$

We set $Z := Z_3 \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \in C^\omega(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$. By the successive definitions of Z_1, Z_2, Z_3, and by Lemma 4.3, for $\varphi \in \mathbb{T}$, we obtain the following expression of the matrix $Z(\varphi)$:

$$Z(\varphi) = \begin{pmatrix} \phi'(\varphi) & \mu(\varphi)\phi'(\varphi) \\ (\phi(\varphi) - \gamma \circ \phi(\varphi))' & (\phi'(\varphi))^{-1} + \mu(\varphi)(\phi(\varphi) - \gamma \circ \phi(\varphi))' \end{pmatrix}$$

(4.14)
$$= \begin{pmatrix} \phi'(\varphi) & \mu(\varphi)\phi'(\varphi) \\ \phi'(\varphi - \alpha) & (\phi'(\varphi))^{-1} + \mu(\varphi)\phi'(\varphi - \alpha) \end{pmatrix}.$$

As ϕ is a circle diffeomorphism, the first coefficient in the matrix does not vanish, and hence the conjugacy map Z is homotopic to the identity. Moreover, for all $\varphi \in \mathbb{T}$, we have

$$Z(\varphi + \alpha)^{-1} S_0^V(\varphi) Z(\varphi) = \begin{pmatrix} 1 & \nu(\varphi) - \mu(\varphi + \alpha) + \mu(\varphi) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \nu_0 \\ 0 & 1 \end{pmatrix}.$$

$\square$

4.4. Pure point spectrum of dual Schrödinger operators.

Since the reduction to the parabolic cocycle (α, B_0) is a key point in the proof of our Main Theorem, we shall provide another proof of this fact based on dual Schrödinger operators and Aubry duality. In general, Aubry duality is based on the fact that the localization properties of the dual Schrödinger operators can be used to show that certain Schrödinger cocycles are reducible. One may consult [7] for more references.

Let $(\widehat{v}_n)_{n\in\mathbb{Z}}$ be the Fourier coefficients of the analytic potential $V := -f' \circ \phi - 2$, i.e., $V\colon \varphi \mapsto \sum_{n\in\mathbb{Z}} \widehat{v}_n e^{2\pi in\varphi}$. For any phase $\varphi_0 \in \mathbb{T}$, we define the *dual Schrödinger operator* $\widehat{H}_{V,\alpha,\varphi_0}$. It acts on $\widehat{u} = (\widehat{u}_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ in the following way:

$$(4.15) \qquad (\widehat{H}_{V,\alpha,\varphi_0}(\widehat{u}))_n := \sum_{k\in\mathbb{Z}} \widehat{v}_{n-k}\widehat{u}_k + 2\cos(2\pi(\varphi_0 + n\alpha))\widehat{u}_n, \quad \forall n \in \mathbb{Z}.$$

Let us denote by $\widehat{\phi'} = (\widehat{\phi'}_n)_{n\in\mathbb{Z}}$ the Fourier coefficients of the function $\phi' \in C^\omega(\mathbb{T}, \mathbb{R})$. In other words, $\phi\colon \varphi \mapsto \sum_{n\in\mathbb{Z}} \widehat{\phi}_n e^{2\pi in\varphi}$, with $\widehat{\phi'}_n = 2\pi in\widehat{\phi}_n$, for $n \in \mathbb{Z}$.

Lemma 4.6. *We have*

$$(4.16) \qquad V(\varphi) = -\frac{\phi'(\varphi + \alpha) + \phi'(\varphi - \alpha)}{\phi'(\varphi)}, \quad \forall \varphi \in \mathbb{T},$$

which yields

$$(4.17) \qquad \widehat{H}_{V,\alpha,0}(\widehat{\phi'}) = 0.$$

By analyticity, the sequence $(\widehat{\phi'}_n)_{n\in\mathbb{Z}}$ decays exponentially fast. In particular, we have $(\widehat{\phi'}_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, and the energy $E = 0$ is in the point spectrum $\Sigma_{\mathrm{pp}}(\widehat{H}_{V,\alpha,0})$ of the dual operator $\widehat{H}_{V,\alpha,0}$.

Proof. By the second identity obtained in (4.11), for all $\varphi \in \mathbb{T}$, it holds

$$f \circ \phi(\varphi) = \phi(\varphi + \alpha) - 2\phi(\varphi) + \phi(\varphi - \alpha),$$

and then, by taking the derivative of the previous expression, we get

$$f' \circ \phi(\varphi) \cdot \phi'(\varphi) = \phi'(\varphi + \alpha) - 2\phi'(\varphi) + \phi'(\varphi - \alpha).$$

Since $V = -f' \circ \phi - 2$, this can also be rewritten as

$$(4.18) \qquad V(\varphi) \cdot \phi'(\varphi) + \phi'(\varphi + \alpha) + \phi'(\varphi - \alpha) = 0,$$

which gives (4.16).

Therefore, in Fourier series, (4.18) yields

$$\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \widehat{v}_{n-k}\widehat{\phi'}_k e^{2\pi in\varphi} + \sum_{n\in\mathbb{Z}} \widehat{\phi'}_n(e^{2\pi in\alpha} + e^{-2\pi in\alpha})e^{2\pi in\varphi} = 0.$$

Equivalently, we have

$$(\widehat{H}_{V,\alpha,0}(\widehat{\phi'}))_n = \sum_{k\in\mathbb{Z}} \widehat{v}_{n-k}\widehat{\phi'}_k + 2\cos(2\pi n\alpha)\widehat{\phi'}_n = 0, \quad \forall n \in \mathbb{Z},$$

which concludes the proof. $\qquad\square$

As in classical Aubry duality, we may then define a Bloch wave $U\colon \varphi \mapsto \begin{pmatrix} \phi'(\varphi) \\ \phi'(\varphi - \alpha) \end{pmatrix}$. It provides an invariant section, in the following sense:[2]

$$(4.19) \qquad S_0^V(\varphi) \cdot U(\varphi) = U(\varphi + \alpha), \quad \forall \varphi \in \mathbb{T}.$$

In our case, the phase is equal to 0 and the frequency α satisfies $\mathfrak{B}(\alpha) < +\infty$, hence by point (2) of the precise version of Aubry-duality given in Avila-Jitomirskaya [7,

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[2]This can also be checked directly using (4.18).

Theorem 2.5], this provides another proof of the reducibility result that we obtained previously in Proposition 4.5.

Note that (4.19) gives another way to see why the connection between existence of analytic invariant curves and parabolic reducibility of Schrödinger cocycles can only happen on the edge of the spectrum: indeed, U is associated to the vector field tangent to the invariant curve, hence should have zero degree. But the latter is directly related to the rotation number of the Schrödinger cocycle (as U can be utilized to reduce the cocycle), which should thus vanish as well; but this corresponds precisely to the right edge of the spectrum.

4.5. **Proof of the Main Theorem.** We remain in the setting of the previous section. Namely, we let $f \in C_0^\omega(\mathbb{T}, \mathbb{R})$ and assume that the twist map $\psi = \psi_f$ leaves invariant an analytic graph Γ_γ, for some function $\gamma \in C^\omega(\mathbb{T}, \mathbb{R})$. We let $g \colon \varphi \mapsto \varphi + \gamma(\varphi) + f(\varphi) \mod 1$ be the induced diffeomorphism, and assume that its rotation number α satisfies the Brjuno condition $\mathfrak{B}(\alpha) < +\infty$. We take $\phi = \phi_\gamma \in C^\omega(\mathbb{T}, \mathbb{T})$ such that $\phi^{-1} \circ g \circ \phi = r_\alpha$, set $V := -f' \circ \phi - 2$, and let $B_0 \in \mathrm{SL}(2, \mathbb{R})$, $Z \in C^\omega(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$ be as in Proposition 4.5. We also let H_{V,α,φ_0} be the Schrödinger operator corresponding to action minimizing trajectories of ψ on Γ_γ.

In this section we will conclude the proof of our Main Theorem on the existence of a component of absolutely continuous spectrum. The idea of the proof is to use Proposition 4.5 together with the openness of the *almost reducibility* property proved by Avila in [3]. Following the arguments of Avila [4], it implies the existence of a component of absolutely continuous spectrum.

Let us recall a few concepts which will be useful in the following.

Given a frequency $\tilde{\alpha} \in \mathbb{R} \setminus \mathbb{Q}$ and a map $A \in C^\omega(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$, the $\mathrm{SL}(2, \mathbb{R})$-cocycle $(\tilde{\alpha}, A)$ is called *subcritical* if there exists $\varepsilon > 0$ such that the associated Lyapunov exponent satisfies $L(\tilde{\alpha}, A(\cdot + \mathrm{i}\delta)) = 0$ for any $|\delta| < \varepsilon$. The cocycle $(\tilde{\alpha}, A)$ is called *almost reducible* if there exists $\varepsilon > 0$ and a sequence $(B^{(n)})_{n \geq 0}$ of maps $B^{(n)} \colon \mathbb{T} \to \mathrm{SL}(2, \mathbb{R})$ admitting holomorphic extensions to the common strip $\{|\Im z| < \varepsilon\}$ such that $B^{(n)}(\cdot + \tilde{\alpha})^{-1} A(\cdot) B^{(n)}(\cdot)$ converges to a constant $\mathrm{SL}(2, \mathbb{R})$-matrix uniformly in $\{|\Im z| < \varepsilon\}$. Let us recall that by Avila's proof of the almost reducibility conjecture (see [3,5,6]), subcriticality implies almost reducibility.

By [7], almost reducibility is related to the notion of *almost localization* which we now recall. For any $x \in \mathbb{R}$, we set $|x|_\mathbb{T} := \inf_{j \in \mathbb{Z}} |x - j|$. Fix $\epsilon_0 > 0$ and $\varphi_0 \in \mathbb{T}$. An integer $k \in \mathbb{Z}$ is called an ϵ_0-*resonance* of φ_0 if $|2\varphi_0 - k\alpha|_\mathbb{T} \leq e^{-\epsilon_0|k|}$ and $|2\varphi_0 - k\alpha|_\mathbb{T} = \min_{|l| \leq |k|} |2\varphi_0 - l\alpha|_\mathbb{T}$.

Definition 4.7 (Almost localization). *Given $\tilde{\alpha} \in \mathbb{R} \setminus \mathbb{Q}$ and $\tilde{V} \in C^\omega(\mathbb{T}, \mathbb{R})$, we say that the family $\{\widehat{H}_{\tilde{V}, \tilde{\alpha}, \varphi}\}_{\varphi \in \mathbb{T}}$ is almost localized if there exist constants $C_0, C_1, \epsilon_0, \epsilon_1 > 0$ such that for all $\varphi_0 \in \mathbb{T}$, any generalized solution $u = (u_k)_{k \in \mathbb{Z}}$ to the eigenvalue problem $\widehat{H}_{\tilde{V}, \tilde{\alpha}, \varphi_0} u = Eu$ with $u_0 = 1$ and $|u_k| \leq 1 + |k|$ satisfies*

$$(4.20) \qquad |u_k| \leq C_1 e^{-\epsilon_1|k|}, \quad \forall C_0|n_j| \leq |k| \leq C_0^{-1}|n_{j+1}|,$$

where $\{n_j\}_j$ is the set of ϵ_0-resonances of φ_0.

By Lemma 2.5 and Proposition 4.5, our Main Theorem is a consequence of the following result.

[Parabolicity $\Rightarrow$ ac] Lemma. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $V \in C^\omega(\mathbb{T}, \mathbb{R})$. Suppose that for some $E_0 \in \mathbb{R}$ the Schrödinger cocycle $(\alpha, S_{E_0}^V)$ is analytically reducible to a constant parabolic cocycle, i.e., there exist $Z \in C^\omega(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$ and $\nu_0 \in \mathbb{R}$ such that*

$$(4.21) \qquad Z(\varphi + \alpha)^{-1} S_{E_0}^V(\varphi) Z(\varphi) = B_0 = \begin{pmatrix} 1 & \nu_0 \\ 0 & 1 \end{pmatrix}, \quad \forall \varphi \in \mathbb{T}.$$

Assume that $\nu_0 < 0$. Then there exists $\varepsilon_0 > 0$ such that for all $\varphi_0 \in \mathbb{T}$,

  (1) *$\Sigma(H_{V,\alpha,\varphi_0}) \cap [E_0 - \varepsilon_0, E_0 + \varepsilon_0] \subset [E_0 - \varepsilon_0, E_0]$;*
  (2) *for any $E \in \Sigma(H_{V,\alpha,\varphi_0}) \cap [E_0 - \varepsilon_0, E_0]$, the Schrödinger cocycle (α, S_E^V) is almost reducible and subcritical;*
  (3) *the restriction of the spectral measures to the interval $[E_0 - \varepsilon_0, E_0]$ is absolutely continuous and positive.*

Here we state the lemma for $\nu_0 < 0$ but of course, a symmetric result holds for $\nu_0 > 0$. It is well known (see for instance [26, 27]) that reducibility at $E = E_0$ to a parabolic matrix different from the identity implies that locally on one side of E_0 the cocycle (α, S_E^V) will be uniformly hyperbolic, and on another side the fibered rotation number will change monotonically (by strict monotonicity of the second iterate of the Schrödinger cocycle with respect to E). In our case, we assume that $\nu_0 < 0$, hence the cocycle (α, S_E^V) is uniformly hyperbolic for $E \in (E_0, E_0 + \varepsilon)$ and its fibered rotation number for $E \in (E_0 - \varepsilon, E_0)$ will be strictly larger than at $E = E_0$. This proves the first statement of the [Parabolicity $\Rightarrow$ ac] Lemma. In the following, we will give the proof of points (2) and (3) in this lemma.

We take $\alpha, V, E_0, Z, B_0, \nu_0$ as in the [Parabolicity $\Rightarrow$ ac] Lemma. To ease the notation, we assume that $E_0 = 0$. Point (3) in the [Parabolicity $\Rightarrow$ ac] Lemma follows from the next result.

Proposition 4.8. *There exists $\varepsilon_1 > 0$ such that for any $E \in (-\varepsilon_1, \varepsilon_1)$, the Schrödinger cocycle (α, S_E^V) is almost reducible, and for any $E \in (-\varepsilon_1, 0) \cap \Sigma(H_{V,\alpha,\varphi_0})$, the cocycle (α, S_E^V) is subcritical.*

Proof. By [3, Corollary 1.3], almost reducibility is an open property in the set of cocycles $\mathbb{R} \setminus \mathbb{Q} \times C^\omega(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$. Hence, we have almost reducibility for all $E \in (-\varepsilon, \varepsilon)$ assuming that $\varepsilon > 0$ is sufficiently small. As recalled above, for positive energies $E \in (0, \varepsilon)$, the cocycle (α, S_E^V) is almost reducible to a hyperbolic $\mathrm{SL}(2, \mathbb{R})$-cocycle (in fact, it is even reducible when $\beta(\alpha) = 0$). Besides, for negative energies $E \in (-\varepsilon, 0) \cap \Sigma(H_{V,\alpha,\varphi_0})$, the cocycle (α, S_E^V) is almost reducible to a parabolic cocycle or a cocycle of rotations.

As the potential V and the conjugacy map Z are analytic, formula (4.21) implies that the cocycle (α, S_0^V) is subcritical. By [6], subcriticality is also an open property in the spectrum (outside of uniform hyperbolicity), which implies the second statement of Proposition 4.8. $\qquad \square$

Given an analytic function $F \in C^\omega(\mathbb{T}, *)$ with $* = \mathbb{R}$ or $\mathfrak{M}_2(\mathbb{R})$, we set $\|F\|_{\mathbb{T}} := \sup_{\varphi \in \mathbb{T}} \|F(\varphi)\|$, and for any $h > 0$ such that F has a bounded analytic extension to the strip $\{|\Im z| < h\}$, we set $\|F\|_h := \sup_{|\Im z| < h} \|F(z)\|$.

In the following, we will give the proof of point (3) in the [Parabolicity $\Rightarrow$ ac] Lemma. Although the result holds for any irrational frequency $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, here we present the proof in the case that $\beta(\alpha) = 0$, as it is the one which is related to our problem about the existence of invariant curves.[3] We refer to [4] for the case $\beta(\alpha) > 0$ (see [4, pp. 16–20]).

It was shown in [7, Theorem 3.2] that there exist absolute constants $c_0, k_0 > 0$ with the following property: for any analytic potential $\tilde{V} \in C^\omega(\mathbb{T}, \mathbb{R})$ such that $\|\tilde{V}_\varepsilon\|_{h_0} < c_0 h_0^{k_0}$ for some $h_0 \in (0, 1]$, the family of dual Schrödinger operators $\{\widehat{H}_{\tilde{V}, \alpha, \varphi_0}\}_{\varphi_0 \in \mathbb{T}}$ is almost localized. In the following, we let $h_0 \in (0, 1)$ be such that the potential V and the conjugacy map Z have a bounded analytic extension to the strip $\{|\Im z| < h_0\}$.

Lemma 4.9. *There exists $\varepsilon_0 \in (0, \varepsilon_1)$ such that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, there exist $\tilde{V}_\varepsilon \in C^\omega(\mathbb{T}, \mathbb{R})$ and $\tilde{Z}_\varepsilon \in C^\omega(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$ such that $\|\tilde{V}_\varepsilon\|_{h_0} < c_0 h_0^{k_0}$,*

$$(4.22) \qquad \tilde{Z}_\varepsilon(\varphi + \alpha)^{-1} S_\varepsilon^V(\varphi) \tilde{Z}_\varepsilon(\varphi) = S_0^{\tilde{V}_\varepsilon}(\varphi), \quad \forall \varphi \in \mathbb{T},$$

and such that the family of dual Schrödinger operators $\{\widehat{H}_{\tilde{V}_\varepsilon, \alpha, \varphi_0}\}_{\varphi_0 \in \mathbb{T}}$ is almost localized.

Proof. Take $\nu_0 < 0$ as above, let $\nu_1 := \sqrt{-\nu_0} > 0$, and set $Q_0 := \begin{pmatrix} -\frac{1}{2}\nu_1 & -\frac{1}{2}\nu_1 \\ \nu_1^{-1} & -\nu_1^{-1} \end{pmatrix} \in$ $\mathrm{SL}(2, \mathbb{R})$. The matrix Q_0 conjugates the parabolic cocycle (α, B_0) to the Schrödinger cocycle $(\alpha, S_0^{\mathbf{0}})$ associated with a vanishing potential $\mathbf{0}$ and the energy $E = 0$, i.e., $Q_0^{-1} B_0 Q_0 = S_0^{\mathbf{0}} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$. Let $Z_0 := Z Q_0 \in C^\omega(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$, so that

$$Z_0(\varphi + \alpha)^{-1} S_0^V(\varphi) Z_0(\varphi) = S_0^{\mathbf{0}} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad \forall \varphi \in \mathbb{T}.$$

For any $\varepsilon \in \mathbb{R}$, we thus obtain

$$Z_0(\varphi + \alpha)^{-1} S_\varepsilon^V(\varphi) Z_0(\varphi) = S_0^{\mathbf{0}} + \varepsilon P_1(\varphi), \quad \forall \varphi \in \mathbb{T},$$

for some analytic map $P_1 \in C^\omega(\mathbb{T}, \mathfrak{M}_2(\mathbb{R}))$. Then, by [8, Lemma 2.2], for any $\tau > 0$, there exists $\delta = \delta(\tau) > 0$ such that if $|\varepsilon| \cdot \|P_1\|_{h_0} < \delta$, then there exists $\tilde{V}_\varepsilon \in C^\omega(\mathbb{T}, \mathbb{R})$ such that $\|\tilde{V}_\varepsilon\|_{h_0} < \tau$, and $\tilde{Z}_\varepsilon \in C^\omega(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$ with a bounded analytic extension to the strip $\{|\Im z| < h_0\}$, such that $\|Z_0 - \tilde{Z}_\varepsilon\|_{h_0} < \tau$, and

$$\tilde{Z}_\varepsilon(\varphi + \alpha)^{-1} S_\varepsilon^V(\varphi) \tilde{Z}_\varepsilon(\varphi) = S_0^{\tilde{V}_\varepsilon}(\varphi), \quad \forall \varphi \in \mathbb{T}.$$

Now, let us take $\tau = \tau_0 := c_0 h_0^{k_0}$ and let $\varepsilon_0 > 0$ be such that $\varepsilon_0 \cdot \|P_1\|_{h_0} < \delta(\tau_0)$. Then, for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the potential $\tilde{V}_\varepsilon$ satisfies $\|\tilde{V}_\varepsilon\|_{h_0} < c_0 h_0^{k_0}$, hence by the definition of c_0, k_0, the family of dual Schrödinger operators $\{\widehat{H}_{\tilde{V}_\varepsilon, \alpha, \varphi_0}\}_{\varphi_0 \in \mathbb{T}}$ is almost localized. $\qquad \square$

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[3]We need $\mathfrak{B}(\alpha) < +\infty$ to guarantee the existence of the conjugacy ϕ to the rigid rotation by angle α.

For any $u \in \ell^2(\mathbb{E})$, any $\varphi_0 \in \mathbb{T}$, we let $\mu^u_{V,\alpha,\varphi}$ be the *spectral measure* of $H = H_{V,\alpha,\varphi_0}$ associated to u, i.e., such that $((H - E)^{-1}u, u) = \int_{\mathbb{R}} \frac{1}{E' - E} d\mu^u_H(E')$, for all $E \in \mathbb{C} \backslash \Sigma$. In what follows, we consider the *canonical spectral measure* corresponding to $u = e_{-1} + e_0$, and denote it by $\mu_{V,\alpha,\varphi} := \mu^{e_{-1}+e_0}_{V,\alpha,\varphi}$. The support of $\mu_{V,\alpha,\varphi}$ is equal to the spectrum $\Sigma(H)$.

Let us also recall the definition of the *integrated density of states* $N_{V,\alpha} \colon \mathbb{R} \to [0, 1]$:

$$N_{V,\alpha}(E) := \int_{\mathbb{T}} \mu_{V,\alpha,\varphi}(-\infty, E] \, d\varphi, \quad \forall E \in \mathbb{R},$$

where for $i \in \mathbb{Z}$, we let $e_i := (\delta_{ij})_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$.

To conclude the proof of the [Parabolicity $\Rightarrow$ ac] Lemma, it remains to show the following.

Proposition 4.10. *The restriction of the spectral measure μ_{V,α,φ_0} to the interval $[-\varepsilon_0, 0]$ is absolutely continuous and positive.*

Proposition 4.10 is proved by repeating the proof of the main result in [4] in the case $\beta(\alpha) = 0$ (see [4, p. 15]). We denote by $\mathcal{B}$ the set of energies $E \in \mathbb{R}$ such that the iterates $(k\alpha, A_k) = (\alpha, S^V_E)^k$ of the cocycle (α, S^V_E) are uniformly bounded, i.e.,

$$\sup_{k \geq 0} \|A_k\|_{\mathbb{T}} < +\infty.$$

Let us recall the following classical characterization (see [18]).

Theorem 4.11. *The restriction $\mu_{V,\alpha,\varphi_0}|_{\mathcal{B}}$ is absolutely continuous for all $\varphi_0 \in \mathbb{R}$.*

In fact, Theorem 4.11 is implied by the following very general estimate contained in [20, 21] and [4].

Lemma 4.12 (Lemma 2.5 in [4]). *For all $\varphi_0 \in \mathbb{T}$, we have*

$$\mu_{V,\alpha,\varphi_0}(E - \epsilon, E + \epsilon) \leq C\epsilon \sup_{0 \leq k \leq C\epsilon^{-1}} \|A_k\|^2_{\mathbb{T}},$$

for some universal constant $C > 0$.

Proof of Proposition 4.10. Let $\Sigma_{\varepsilon_0} := \Sigma(H_{V,\alpha,\varphi_0}) \cap [-\varepsilon_0, 0]$, let $\mathcal{B}$ be the set of energies $E \in \Sigma_{\varepsilon_0}$ such that the cocycle (α, S^V_E) is bounded, and let $\mathcal{R}$ be the set of energies $E \in \Sigma_{\varepsilon_0}$ such that (α, S^V_E) is reducible. By Theorem 4.11 recalled above, it is sufficient to prove that for every $\varphi_0 \in \mathbb{T}$, the canonical spectral measure $\mu = \mu_{V,\alpha,\varphi_0}$ satisfies $\mu(\Sigma_{\varepsilon_0} \backslash \mathcal{B}) = 0$.

As noted in [4], for any $E \in \mathcal{R} \backslash \mathcal{B}$, we have $N_{V,\alpha}(E) \in \mathbb{Z} \oplus \alpha \mathbb{Z}$, and (α, S^V_E) is analytically reducible to a parabolic cocycle; in particular, $\mathcal{R} \backslash \mathcal{B}$ is countable. Moreover, there are no eigenvalues in $\mathcal{R}$ (if $H_{V,\alpha,\varphi_0}u = Eu$ with $E \in \mathcal{R}$ and $u \neq 0$, then $\inf_{n \in \mathbb{Z}} |u_n|^2 + |u_{n+1}|^2 > 0$ hence $u \notin \ell^2(\mathbb{Z})$), and then, $\mu(\mathcal{R} \backslash \mathcal{B}) = 0$. Therefore, it is enough to prove that $\mu(\Sigma_{\varepsilon_0} \backslash \mathcal{R}) = 0$.

By (4.22), for any energy $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the cocycle $(\alpha, S^V_\varepsilon)$ is (almost) reducible if and only the cocycle $(\alpha, S^{\tilde{V}_\varepsilon}_0)$ is. Moreover, by Theorem 2.5 and Theorem 4.1 in [7], (almost) reducibility of $(\alpha, S^{\tilde{V}_\varepsilon}_0)$ is related to the (almost) localization of the family of dual Schrödinger operators $\{\widehat{H}_{\tilde{V}_\varepsilon,\alpha,\varphi}\}_{\varphi \in \mathbb{T}}$. We know by [7, Theorem 3.3]

that there exist a phase $\varphi_0 \in \mathbb{T}$ and a sequence $\widehat{u} = (\widehat{u}_j)_{j \in \mathbb{Z}}$, with $\widehat{u}_0 = 1$ and $|\widehat{u}_j| \leq 1$ for all $j \in \mathbb{Z}$, such that $\widehat{H}_{\tilde{V}_\varepsilon, \alpha, \varphi_0} \widehat{u} = 0$. As explained in [7], $\widehat{u}$ can be utilized to (almost) reduce the cocycle $(\alpha, S_0^{\tilde{V}_\varepsilon})$. By almost localization of $\widehat{H}_{\tilde{V}_\varepsilon, \alpha, \varphi_0}$, $\widehat{u}$ decays exponentially fast between the resonances as in (4.20). If φ_0 is not resonant, then by [7, Remark 3.3], the sequence $(\widehat{u}_j)_{j \in \mathbb{Z}}$ decays exponentially fast and $(\alpha, S_0^{\tilde{V}_\varepsilon})$ is actually reducible. In the following, we consider the case where we have resonances.

Following [4], for any integer $m \geq 0$, we let $K_m \subset \Sigma_{\varepsilon_0}$ be the set of energies E such that for some $\varphi_0 \in \mathbb{T}$, the dual operator $\widehat{H} = \widehat{H}_{\tilde{V}_E, \alpha, \varphi_0}$ has a bounded normalized solution $\widehat{H}\widehat{u} = 0$ with a resonance $2^m \leq |n_j| < 2^{m+1}$. By the Borel-Cantelli lemma, to conclude the proof, it is enough to show that $\sum_m \mu(\overline{K_m}) < +\infty$. Indeed, by Theorem 3.3 in [4], we have $\Sigma_{\varepsilon_0} \setminus \mathcal{R} \subset \limsup_m K_m$, and then, $\sum_m \mu(\overline{K_m}) < +\infty$ implies that $\mu(\Sigma_{\varepsilon_0} \setminus \mathcal{R}) \leq \mu(\limsup_m K_m) = 0$.

By [4, Theorem 3.8], there exist constants $C_1, c_1 > 0$ such that for each integer $m \geq 0$ and each energy $E \in K_m$, there exists an open neighborhood $J_m(E)$ of E of size $\epsilon_m := C_1 e^{-c_1 2^m}$ so that the iterates of $(\alpha, A) = (\alpha, S_E^V)$ satisfy $\sup_{0 \leq k \leq 10\epsilon_m^{-1}} \|A_k\|_{\mathbb{T}} \leq e^{o(2^m)}$. By Lemma 4.12, we thus get

$$(4.23) \qquad \mu(J_m(E)) \leq C e^{o(2^m)} |J_m(E)|,$$

where $|\cdot|$ is the Lebesgue measure. Take a finite subcover $\overline{K_m} \subset \cup_{j=0}^{r_m} J_m(E_j)$ such that every $x \in \mathbb{R}$ is contained in at most 2 different $J_m(E_j)$.

By [4, Lemma 3.11], the integrated density of states $N_{V,\alpha}$ satisfies $|N_{V,\alpha}(J_m(E))| \geq c|J_m(E)|^2$, for some constant $c > 0$. Besides, by [4, Lemma 3.13], there exist constants $C_2, c_2 > 0$ such that for any $E \in K_m$, it holds $\|N_{V,\alpha}(E) - k\alpha\|_{\mathbb{T}} \leq C_2 e^{-c_2 2^m}$ for some integer k with $|k| < C_2 2^m$. Therefore, the set $N_{V,\alpha}(K_m)$ can be covered by $C_2 2^{m+1}$ intervals $(T_m^k)_{0 \leq k \leq C_2 2^{m+1}}$ of length $C_2 e^{-c_2 2^m}$. For some constant $C_3 > 0$, we have $|T_m^k| < C_3 |N_{V,\alpha}(J_m(E))|$, for any integers $m \geq 0$, $0 \leq k \leq C_2 2^{m+1}$, and any energy $E \in K_m$. Hence, for a given interval T_m^k, there are at most $2C_3 + 4$ intervals $J_m(E_j)$ such that $N_{V,\alpha}(J_m(E_j))$ intersects T_m^k. We deduce that for each integer $m \geq 0$, it holds $r_m \leq C_4 2^m$, with $C_4 := 4C_2(C_3 + 2)$. Then, by (4.23), we obtain

$$\mu(K_m) \leq \sum_{j=0}^{r_m} \mu(J_m(E_j)) \leq C_4 2^m \cdot C e^{o(2^m)} \cdot C_1 e^{-c_1 2^m} = O(e^{-c_1 2^{m-1}}),$$

which gives $\sum_m \mu(\overline{K_m}) < +\infty$, and concludes the proof of Proposition 4.10. $\qquad\square$

As a consequence of point (2) in the [Parabolicity $\Rightarrow$ ac] Lemma and of [24, Theorem 7.1] for a frequency α satisfying the weak Diophantine condition $\beta(\alpha) = 0$ and in the subcritical regime, we also get some result about the homogeneity of the spectrum near its right edge:

Proposition 4.13. *Under the same assumptions as in the* [Parabolicity $\Rightarrow$ ac] *Lemma, if α moreover satisfies $\beta(\alpha) = 0$, then there exists $\kappa > 0$ such that $|(E - \varepsilon, E + \varepsilon) \cap \Sigma(H_{V,\alpha,\varphi_0})| > \kappa\varepsilon$, for all $E \in \Sigma(H_{V,\alpha,\varphi_0}) \cap (E_0 - \varepsilon_0, E_0)$, and for all $0 < \varepsilon < E_0 - E$.*

5. Concluding remarks

The main result of this paper is the existence of a component of absolutely continuous spectrum whenever there exists an analytic invariant curve. This is a semi-global result, which does not require explicitly the smallness of the potential. We finish this paper with several conjectures and questions related to the effect that the transitions from KAM to weak KAM regime has on the spectrum of the corresponding Schrödinger operators.

As in the introduction we consider a one-parameter family of twist maps $\{\psi_{\lambda f}\}_{\lambda \in \mathbb{R}}$. For a fixed typical rotation number α it is believed that

(1) for $0 \leq \lambda < \lambda_{cr}(\alpha)$, there exists a smooth invariant curve such that the restricted dynamics is conjugated to a rigid rotation by α;

(2) for $\lambda = \lambda_{cr}(\alpha)$, the invariant curve still exists but it is not analytic anymore (the *critical curve*, possibly, is only $C^{1+\epsilon}$-smooth);

(3) for $\lambda > \lambda_{cr}(\alpha)$, there exists an invariant cantori with rotation number α. The dynamics on the cantori is hyperbolic.

Below, we discuss how the above dynamical transition from elliptic dynamics (KAM) to hyperbolic dynamics (weak KAM) may affect the spectral type of the corresponding Schrödinger operators.

It is likely that in the weak KAM regime $\lambda > \lambda_{cr}(\alpha)$ there will be no absolutely continuous spectrum. This case corresponds to rather rough discontinuous potentials. The case where the potential has just one jump discontinuity was considered in [11]. It was shown there that the spectrum has no absolutely continuous component. The proof is based on the non-deterministic argument and Kotani approach which imply that the set of energies at which the Lyapunov exponent vanishes has zero Lebesgue measure. In the weak KAM case the Kotani theorem should be extended in the direction of asymptotic non-determinism. It follows from the hyperbolicity of the dynamics that one can find two different sequences $\{V_0^{1,2}(n), n \in \mathbb{Z}\}$ of values of potentials which exponentially converge to each other as $n \to \pm\infty$. It is natural to ask whether the Kotani argument can be extended to this case.

We have shown that an absolutely continuous component exists for $\lambda < \lambda_{cr}(\alpha)$. It is an interesting question whether the spectrum is mixed. Since dynamical properties are related only to the edge of the spectrum it is natural to expect coexistence of absolutely continuous spectrum and point spectrum for $0 \leq \lambda_{cr}(\alpha) - \lambda \ll 1$.

As was said above we do not expect absolutely continuous spectrum in the super-critical case $\lambda > \lambda_{cr}(\alpha)$. Again one can ask whether the spectrum is mixed in this case. Notice that for $\lambda > \lambda_{cr}(\alpha)$ the edge of the spectrum is expected to move to the left. Namely, $\Sigma(\mathcal{H}) \subset (-\infty, E(\lambda)]$ with $E(\lambda) < 0$, and $E(\lambda) \to 0$ as $\lambda \to \lambda_{cr}(\alpha)$.

Finally, it is natural to ask what is the spectral type near the edge of the spectrum $E = 0$ in the critical case $\lambda = \lambda_{cr}(\alpha)$. It is tempting to think that there a singular continuous component is created.

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