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ANDERSON LOCALISATION IN STATIONARY ENSEMBLES OF QUASIPERIODIC OPERATORS

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Abstract. An ensemble of quasi-periodic discrete Schrödinger operators with an arbitrary number of basic frequencies is considered, in a lattice of arbitrary dimension, in which the hull function is a realisation of a stationary Gaussian process on the torus. We show that, for almost every element of the ensemble, the quasi-periodic operator boasts Anderson localization with simple pure point spectrum at strong coupling. One of the ingredients of the proof is a new lower bound on the interpolation error for stationary Gaussian processes on the torus (also known as local non-determinism).

1. Introduction

We consider quasiperiodic Schrödinger operators on $\mathbb{Z}^d$ (equipped with the graph metric $\|\cdot\|$), for arbitrary $d \geq 1$ and an arbitrary number of frequencies $\nu \geq 1$. Let $\mathbb{T}^\nu = (\mathbb{R}/\mathbb{Z})^\nu$; fix a continuous function $v : \mathbb{T}^\nu \to \mathbb{R}$, a $\nu \times d$ frequency matrix $\alpha = (\alpha_{ij})$, an initial point $\omega \in \mathbb{T}^\nu$, and a coupling $g > 0$, and define an operator $H = H(\omega; g)$ on $\ell_2(\mathbb{Z}^d)$ by

$$(H(\omega; g)f)(x) = \sum_{\|y-x\|=1} f(y) + gv(\omega + \alpha x)f(x) \ ,. \tag{1.1}$$

Operators of the form $H(\omega; g)$ form an important subclass of metrically transitive (ergodic) operators [39].

Operators of the form (1.1) have been intensively studied for $d = \nu = 1$. It was found that for large $g \geq g_0$ and Diophantine α, the operator exhibits Anderson localisation, manifesting itself in pure point spectrum with exponentially decaying eigenfunctions. This phenomenon has been rigorously established first for the Maryland model $v(\omega) = \tan(2\pi\omega)$ and for more general tangent-like potentials [5, 36, 38, 42] (following the physical work [20]), then for the Almost Mathieu model $v(\omega) = \cos(2\pi\omega)$ and more general cosine-like potentials [22, 25, 26, 43], and, more recently, for general analytic potentials [8, 9] and further for potentials in Gevrey classes [30, 31]. We refer to the survey [37] for a review of the state of art. In [10], Anderson localisation was established for a class of analytic potentials for $d = 1$ and $\nu = 1, 2$.

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Much less is known for $d > 1$. The analysis of tangent-like potentials was extended to higher dimension in [5]. In [16], quasiperiodic potentials exhibiting pure point spectrum were constructed using an inverse spectral procedure. In [11], Anderson localisation at strong coupling was proved for analytic potentials and $d = \nu = 2$; this result is perturbative, meaning that for each ω localisation holds outside a set of frequencies the measure of which tends to zero as $g \to \infty$. In [12], the result of [11] was extended to arbitrary $d = \nu$. We also mention the work [28] on delocalisation, i.e. the existence of absolutely continuous spectrum, at weak coupling (for an operator in the continuum).

These results raised the question whether Anderson localisation persists when v is less smooth, e.g. has a finite number of derivatives. Another question is whether localisation holds in the non-perturbative setting for $d > 1$, under a usual Diophantine condition on the frequency. As these questions are yet to be answered for explicit v such as $v(\theta) = \sum_j \cos\theta_j$, it was suggested in [14, 15] to study the properties of (1.1) for typical hull functions v: namely, v is chosen as a realisation of a stochastic process on $\mathbb{T}^\nu$. Related ideas appeared in the work [13]. In these works, Anderson localisation was established for v sampled from a class of (non-stationary) stochastic processes, constructed to ensure the required properties. Here, we extend these results to the more natural class of stationary Gaussian processes on the torus:

$$v(\omega) = \sum_{\ell \in (2\pi\mathbb{Z})^\nu} \frac{g_l \cos\langle\omega, \ell\rangle + h_\ell \sin\langle\omega, \ell\rangle}{\sqrt{W(\ell)}} \ , \quad \omega \in \mathbb{T}^\nu \ , \tag{1.2}$$

where g_ℓ and h_ℓ are jointly independent standard Gaussian random variables, and $W : 2\pi\mathbb{Z}^\nu \to \mathbb{R}_+$ is a spectral weight. Denote the underlying probability space by $(\Theta, \mathcal{B}^\Theta, \mathbb{P}^\Theta)$; to emphasise the dependence on θ, we write $v(\omega) = v(\omega, \theta)$. Denote the operator corresponding to $\theta \in \Theta$ by $H(\omega, \theta; g)$.

Theorem 1.1. *Assume that* $W : 2\pi\mathbb{Z}^\nu \to \mathbb{R}_+$ *is such that*

$$c\|\ell\|^{\nu+\delta} \le W(\ell) \le Ce^{C\|\ell\|^\zeta} \ , \quad \ell \in 2\pi\mathbb{Z}^\nu \ ,$$

for some $\kappa, \zeta, \delta > 0$*, and* $C, c > 0$*, and that* α *satisfies the Diophantine condition*

$$\operatorname{dist}(\alpha x, \mathbb{Z}^\nu) \ge c'\|x\|^{-A} \ , \quad x \in \mathbb{Z}^d \setminus \{0\} \tag{1.3}$$

with some $A > 0$ *and* $c' > 0$*. If* $(A+1)\zeta < 1$*, then there exists a map* $\Theta^+ : \mathbb{R}_+ \to \mathcal{B}^\Theta$ *such that* $\mathbb{P}^\Theta(\Theta^+(g)) \to 1$ *as* $g \to +\infty$*, and for every* $\theta \in \Theta^+(g)$ *and almost every* $\omega \in \mathbb{T}^\nu$*, the spectrum of the operator* $H(\omega, \theta; g)$ *constructed from (1.2) is pure point, and every eigenfunction* ψ *of* $H(\omega, \theta; g)$ *satisfies*

$$\sup_{x \in \mathbb{Z}^d} |\psi(x)| e^{\|x\|} < \infty \ . \tag{1.4}$$

Remark 1.2. According to a theorem of Groshev [6, 23], for α in a set of full measure the condition (1.3) holds with any $A > d/\nu$.

Remark 1.3. As part of the proof, we show in Lemma 2.11 that the number of "resonances" is uniformly bounded. For processes with uniformly Lipschitz realisation, our uniform bound $k_{\max} = \nu + 1$ is optimal, as $\nu + 1$-fold resonances are known to be topologically unavoidable. For a different class of Gaussian processes, the same conclusion was established in [14].

The main theorem follows from two propositions. The first one, Proposition 1.4, establishes the conclusion of Theorem 1.1 in a more abstract setting, when $\omega+\alpha x$ in (1.1) is replaced with an orbit of an ergodic action of $\mathbb{Z}^d$ on a metric probability space Ω. The second one, Proposition 1.6, confirms that the assumptions are satisfied for the process (1.2).

A general localisation theorem. In this section, we replace the torus $\mathbb{T}^\nu$ with a metric probability space $(\Omega, \mathcal{B}^\Omega, \mathbb{P}^\Omega, \mathrm{dist})$ of finite metric dimension, i.e. we assume that there exists $\nu > 0$ (not necessarily integer) such that, for any $\epsilon \in (0,1]$, Ω admits an ϵ-net of cardinality at most $(C/\epsilon)^\nu$. Let $T : \Omega \times \mathbb{Z}^d \to \Omega$ be an ergodic action of $\mathbb{Z}^d$ on Ω satisfying the Diophantine property

$$(1.5) \qquad \mathbf{(UPA)}_A \quad \inf_\omega \min_{0<\|x\|\le L} \mathrm{dist}(T^x\omega,\omega) \ge cL^{-A}\ , \quad L \in \mathbb{N}\ .$$

For the case of $\mathbb{T}^\nu$ with the action $T^x\omega = \omega + \alpha x$, the condition $\mathbf{(UPA)}_A$ boils down to the Diophantine property (1.3).

Let $(\Theta, \mathbb{B}^\Theta, \mathbb{P}^\Theta)$ be an additional probability space, and let $v(\omega,\theta)$ be a (modification of a) stochastic process defined on Θ and taking values in the space of uniformly κ-Hölder-continuous functions from Ω to $\mathbb{R}$ (for some fixed $\kappa > 0$), so that for any $\omega \in \Omega$ the conditional distribution of the random variable $v(\omega,\cdot)$ conditioned on the values of the process in the complement to the ϵ-neighbourhood $Q_\epsilon(\omega)$ of ω is absolutely continuous and admits a density satisfying the local interpolation bound

$$(1.6) \qquad \mathbf{(LIB)}_\eta \quad p_\omega(t \mid \Omega \setminus Q_\epsilon(\omega)) \le \exp(C\epsilon^{-\eta})\ , \quad \epsilon \in (0,\epsilon_0]\ ,$$

Here and below, $p_\omega(t \mid \mathcal{A})$ with $\mathcal{A} \subset \Omega$ denotes the conditional probability density of the random variable $v(\omega,\cdot)$ on Θ conditioned on the sub-sample $\{v(\omega',\cdot),\, \omega' \in \mathcal{A}\}$; a similar notation is used, e.g., in (1.10) for the conditional variance. Then we replace (1.1) with the more general metrically transitive operator

$$(1.7) \qquad (H(\omega,\theta;g)f)(x) = \sum_{\|y-x\|=1} f(y) + gv(T^x\omega,\theta)f(x)\ .$$

Proposition 1.4. *Assume that the assumptions* $\mathbf{(UPA)}_A$ *and* $\mathbf{(LIB)}_\eta$ *hold with* A *and* η *such that* $A\eta < 1$. *Then there exists a map* $\Theta^+ : \mathbb{R}_+ \to \mathcal{B}^\Theta$ *such that* $\mathbb{P}^\Theta(\Theta^+(g)) \to 1$ *as* $g \to +\infty$, *and for every* $\theta \in \Theta^+(g)$ *and almost every* $\omega \in \Omega$, *the spectrum of the operator* $H(\omega,\theta;g)$ *is pure point, and every eigenfunction* ψ *satisfies*

$$(1.8) \qquad \sup_x |\psi(x)| e^{\|x\|} < \infty\ .$$

Remark 1.5. Proposition 1.4 (and, accordingly, also Theorem 1.1) can be strengthened in several directions, without invoking new methods:

(1) the rate of exponential decay (1.4) can be improved to $\sup_x |\psi(x)| e^{m_g\|x\|} < \infty$ for an arbitrary $m_g = o(g)$;
(2) on the event $\Theta^+(g)$, the operator can be shown to exhibit dynamical localisation (our bounds on the eigenfunctions are sufficient to control the eigenfunction correlators [1–3]);
(3) on the event $\Theta^+(g)$, the spectrum of H can be shown to be simple (see [15], building on the method of [29]).

Interpolation of stationary processes. Consider a stationary Gaussian process

$$v(\omega) = \sum_{\ell \in (2\pi\mathbb{Z})^\nu} \frac{g_\ell \cos\langle \omega, \ell \rangle + h_\ell \sin\langle \omega, \ell\rangle}{\sqrt{W(\ell)}} , \quad \omega \in \mathbb{T}^\nu , \tag{1.9}$$

as in (1.2). For $0 < \epsilon \leq 1/2$ let

$$\mathbf{V}(\epsilon) = \operatorname{Var}\left(v(\omega) \,\middle|\, \{v(\omega') \,:\, \omega' \in \mathbb{T}^\nu \,,\, \|\omega' - \omega\| \geq \epsilon\}\right) \tag{1.10}$$

be the conditional variance of $v(\omega)$ conditioned on the complement to the ϵ-neighbourhood of ω (here and forth $\|\cdot\| = \|\cdot\|_\infty$ is the ℓ_∞ distance from 0 on $\mathbb{T}^\nu$).

Proposition 1.6. *Assume that there exists a non-decreasing function $M : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\int_{t_0}^\infty \frac{\log M(t)}{t^2} dt < \infty , \quad K = \sum_{\ell \in 2\pi\mathbb{Z}^\nu} \frac{W(\ell)}{M(\|\ell\|)} < \infty . \tag{1.11}$$

Then for

$$0 < \epsilon \leq \min\left(\frac{1}{2}, \frac{e}{2}\int_0^\infty \frac{\log M(t)}{t^2} dt\right)$$

the conditional variance $\mathbf{V}(\epsilon)$ admits the lower bound

$$\mathbf{V}(\epsilon) \geq \frac{1}{C_\nu K \epsilon^{2\nu} M(S^{-1}(\frac{2}{e}\epsilon))} , \quad \textit{where} \quad S(t) = \int_t^\infty \frac{\log M(\tau)}{\tau^2} d\tau , \quad C_\nu = e^2 2^\nu .$$

Remark 1.7. The asymptotic behaviour of $\mathbf{V}(\epsilon)$ as $\epsilon \to +0$ is an aspect of the interpolation problem for stationary Gaussian processes, going back to [32]. The interpolation problem was studied, for the $\nu = 1$ case of the full-space process

$$\tilde{v}(\xi) = \int_{\mathbb{R}^\nu} \frac{\cos\langle \xi, \lambda\rangle dB_1(\lambda) + \sin\langle \xi, \lambda \rangle dB_2(\lambda)}{\sqrt{(2\pi)^d W(\lambda)}} , \quad \xi \in \mathbb{R}^\nu , \tag{1.12}$$

in [19, §4.13 and Ch. 6] (where B_1 and B_2 are Brownian motions). The connection with the theory of de Branges spaces and Krein strings, established in these works, allows, in particular, to compute $\mathbf{V}(\epsilon)$ explicitly in several examples. A condition of the form (1.11) is unavoidable: for sufficiently regular weights W, it holds for an appropriately chosen majorant M whenever $V(\epsilon) \not\equiv 0$.

Quantitative bounds for $\mathbf{V}(\epsilon)$ in the $\nu = 1$ case of (1.12) were obtained by [18], building on the work [17]. When applied to (1.12), our method yields marginally weaker bounds for $W(\lambda) \propto |\lambda|^\alpha$ and marginally stronger ones for any faster-growing W, particularly, for $W(\lambda) \propto \exp(\|\lambda\|^\zeta)$. Another advantage is that our estimate is somewhat more explicit, and adjusts easily to the process on the torus $\mathbb{T}^\nu$ (for arbitrary ν), as is required here. On the other hand, it is conceivable that a bound sufficient for Theorem 1.1 can be also obtained by the method of [18].

Proof of Theorem 1.1. Assume that

$$c\|\ell\|^{\nu+\delta} \leq W(\ell) \leq C\exp(C\|\ell\|^\zeta) .$$

Fix $0 < \kappa < \delta$; the lower bound ensures that the realisations of v are almost surely uniformly κ-Hölder continuous. From the upper bound,

$$\sum_{\ell} \frac{W(\ell)}{M(\|\ell\|)} < \infty \ , \quad \text{where} \quad M(t) = e^{2Ct^{\zeta}} \ .$$

We apply Proposition 1.6:

$$S(t) = \int_t^{\infty} \frac{2C\tau^{\zeta}}{\tau^2} d\tau \leq C_1 t^{-(1-\zeta)} \ , \quad S^{-1}(\epsilon) \leq C_2 \epsilon^{-\frac{1}{1-\zeta}} \ ,$$

therefore

$$\mathbf{V}(\epsilon) \geq \frac{1}{C_3 \exp(C_4 \epsilon^{-\frac{\zeta}{1-\zeta}})} \ ,$$

i.e. $\mathbf{(LIB)}_{\eta}$ holds with $\eta = \zeta/(1-\zeta)$. The assumption $\zeta(A+1) < 1$ ensures that $\eta A < 1$, hence we can apply Proposition 1.4. ☐

2. Multiscale analysis: Proof of Proposition 1.4

The proof of Proposition 1.4 is based on multi-scale analysis, originating in the work [21] on random operators. Our version of the argument, building on [14, 15], is organised as follows: a deterministic inductive procedure is established in Proposition 2.4 of Section 2.1, and then, in Section 2.2, we verify that the conditions of Proposition 2.4 are satisfied for our random operator (on an event of full probability). The main technical difference compared to the works [14, 15] is the use of $2L \times L$ rectangles (and more generally $2L \times L \times \cdots \times L$ cuboids) instead of squares and cubes in the induction.

2.1. **Scale induction.** In this section, H is a fixed discrete Schrödinger operator acting on $\ell_2(\mathbb{Z}^d)$. For a finite $B \subset \mathbb{Z}^d$, denote by H_B the restriction of H to B, i.e. $H_B = P_B H P_B^*$, where $P_B : \ell_2(\mathbb{Z}^d) \to \ell_2(B)$ is the coordinate projection. For $E \in \mathbb{R}$, let $G_E[H_B] = (H_B - E)^{-1}$ be the resolvent of H_B at E.

The multi-scale induction involves the parameters $m > 0$, $b \in (0,1)$, $\gamma \in (2-b, \infty)$ and $J \in \mathbb{N}$, which will be fixed throughout the argument (that is, one may choose them tailored to the operator H). Their rôles are as follows:

- m is a "mass", controlling the rate of exponential decay of the Green function in infinite volume;
- b is responsible for the deterioration of the mass: on the scale L, the mass will be $m(1 + L^{-(1-b)})$;
- γ is responsible for the growth of scales: we fix L_0 (the scale of the box used as the induction base), and let $L_{k+1} = \lfloor L_k^{\gamma} \rfloor$;
- $J \geq 1$ controls the number of "resonances".

Definition 2.1. A box is a product of d intervals: $B = I_1 \times \cdots \times I_d \subset \mathbb{Z}^d$. We denote by $\mathfrak{B}$ the collection of all boxes, and by $\mathfrak{B}_2$ the collection of sets $b_1 \setminus b_2$, where b_1, b_2 are boxes.

A box $R \subset \mathbb{Z}^d$ is called an L-rectangle if $d-1$ of the intervals in the product are of cardinality $2L+1$ (i.e. of length $2L$) and one is of cardinality $L+1$ (i.e. of length L).

The boundary of $s \subset \mathbb{Z}^d$ is the set $\partial s \subset \mathbb{Z}^d \times \mathbb{Z}^d$ of pairs $(u, u') \in s \times (\mathbb{Z}^d \setminus s)$ such that $\|u - u'\| = 1$. The projection of ∂s onto the first coordinate is denoted $\partial_{\text{in}} s (\subset s)$.

Definition 2.2. Given $E \in \mathbb{R}$, an L-rectangle R is called E-regular if

$$\forall x, y \in \partial_{\text{in}} R \quad \text{s.t. } \|x - y\| \geq L : \quad |G_E[H_R](x, y)| \leq e^{-m(L+L^b)} \ . \tag{2.1}$$

Otherwise, R is called E-singular.

A set $B \subset \mathbb{Z}^d$ is called (E, L)-resonant if there exists $s \in \mathcal{B}_2 \cap 2^B$ such that $\|G_E[H_s]\| > \exp(\frac{mL^b}{16J})$; otherwise, B is called (E, L)-nonresonant.

Definition 2.3. Let $J \geq 1$. A collection $\mathfrak{S} \subset 2^{\mathbb{Z}^d} \setminus \{\varnothing\}$ is said to be J-sparse in $B \subset \mathbb{Z}^d$ if $\mathfrak{S} \cap 2^B$ does not contain J pairwise disjoint sets. We colloquially write, for example, "E-resonant L-rectangles are 2-sparse in s" as a shorthand for "the collection of all E-resonant L-rectangles is 2-sparse in the set s".

Proposition 2.4. *For any $m > 0$, $b \in (0, 1)$, $\gamma \in (2 - b, \infty)$ and $J \geq 1$ there exists $L_* = L_*(m, b, \gamma, J, d)$ such that the following holds whenever $L_0 \geq L_*$. Assume that for any $E \subset \mathbb{R}$*

(1) *for any $k \geq 0$, (E, L_k)-resonant L_{k+1}-rectangles are J-sparse in any L_{k+2}-rectangle, and 2-sparse in the box $[-L_{k+2}, L_{k+2}]^d$;*
(2) *E-singular L_0-rectangles are J-sparse in any L_1 rectangle.*

Then

(a) *the spectrum of H is pure point;*
(b) *for any eigenfunction ψ, $\sup_x |\psi(x)| \exp(\frac{m}{16}\|x\|) < \infty$.*

Remark 2.5. The denominator 16 in (b) can be replaced with any number greater than 1.

In this section we prove Proposition 2.4, which will be derived from

Proposition 2.6. *For any $m > 0$, $b \in (0, 1)$ and $J \geq 1$ the following holds for $L \geq L_*(m, b, J, d)$. Fix $E \in \mathbb{R}$, and suppose R' is an L'-rectangle such that*

(1) *E-singular L-rectangles are J-sparse in R';*
(2) *R' is (E, L)-nonresonant;*
(3) *$L \leq L' \leq \exp(\frac{mL^b}{100dJ})$.*

Then

(a) *for any $x, y \in R'$ with $\|x - y\| \geq 4JL$*

$$|G_E[H_{R'}](x, y)| \leq e^{-\frac{m}{2}\|x-y\|} \ ; \tag{2.2}$$

(b) *if $100JL^{2-b} \leq L' \leq \exp(\frac{mL^b}{100\nu J})$, then R' is E-regular.*

Proof of Proposition 2.4. First, we fix E and prove by induction that, for any $k \geq 0$, E-singular L_k-rectangles are J-sparse in any L_{k+1}-rectangle. By the second assumption, this property holds for $k = 0$. Assume that the property holds for some k and fails for $k+1$. Then there is an L_{k+2}-rectangle R'' containing J disjoint singular L_{k+1}-rectangles R'_j, $j = 1, \cdots, J$. By the induction hypothesis, E-singular L_k-rectangles are J-sparse in each of the R'_j. By the first assumption, at least one of them, say, R'_1, is (E, L_k)-nonresonant. Also, if L_0 is large enough, then $L = L_k$ and $L' = L_{k+1} = \lfloor L^\gamma \rfloor$ satisfy the inequalities

$$100JL^{2-b} \leq L' \leq \exp(\frac{mL^b}{100dJ}) \ .$$

Thus R'_1 satisfies all the conditions of part (b) of Proposition 2.6, and is therefore E-regular, in contradiction to our assumption.

Second, we show that for any E and $k \geq 0$, and any (E, L_k)-nonresonant L_{k+1} rectangle R',

(2.3)
$$\forall x, y \in R' : \quad \Big(\|x - y\| \geq 4JL_k \implies |G_E[H_{R'}](x, y)| \leq \exp(-\frac{m}{2}\|x - y\|) \Big) \ .$$

This follows from part (a) of Proposition 2.6, using the first step of the current proof to verify the first condition of the proposition.

Now we are in position to prove the proposition. Schnol's lemma [7] implies that for almost any E with respect to the spectral measure of H there exists a non-trivial formal solution ψ of the eigenfunction equation $H\psi = E\psi$ such that $|\psi(x)| \leq (\|x\| + 1)^d$. By the first assumption, (E, L_k)-resonant L_{k+1}-rectangles are 2-sparse in the box $[-L^d_{k+2}, L^d_{k+2}]$. By the second step of the current proof, any (E, L_k)-nonresonant L_{k+1}-rectangle R' satisfies (2.3), hence for any point $x \in R'$ with $\mathrm{dist}(x, \partial_{\mathrm{in}} R') \geq 4JL_k$

$$(2.4) \qquad \begin{aligned} |\psi(x)| &\leq \sum_{uu' \in \partial R'} |G_E[H_{R'}](x, u)||\psi(u')| \\ &\leq (3L_{k+1})^d e^{-2mJL_k} (1 + L_{k+2})^d \leq e^{-mJL_k} \ . \end{aligned}$$

The right-hand side of (2.4) tends to zero as $k \to \infty$. Fix a point x_* such that $\psi(x_*) \neq 0$, then for $k \geq k_0 = k_0(x_*)$ the inequality has to fail, i.e. every L_{k+1}-rectangle $R' \ni x_*$ such that $\mathrm{dist}(x_*, \partial_{\mathrm{in}} R') \geq 4JL_k$ has to be (E, L_k)-resonant.[1]

Let $\tilde{R}' \subset [-L_{k+2}, L_{k+2}]^d \setminus [x_* - 4JL_k, x_* + 4JL_k]^d$ be an L_{k+1}-rectangle. Then there exists an L_{k+1}-rectangle R' disjoint from $\tilde{R}'$ such that $R' \ni x_*$ and $\mathrm{dist}(x_*, \partial_{\mathrm{in}} R') \geq 4JL_k$. As R' is (E, L_k)-resonant, we conclude that $\tilde{R}'$ is (E, L_k)-nonresonant. This implies that

$$(2.5) \qquad \forall k \geq k_0(x_*) \ \forall x \ \big(\|x\| \in [8JL_k, L_{k+2} - 3L_{k+1}] \implies |\psi(x)| \leq e^{-mJL_k} \big) \ .$$

In particular, ψ lies in $\ell_2(\mathbb{Z}^d)$. This holds for every ψ, hence the spectrum of H is pure point.

Consider the function $\phi(x) = |\psi(x)| e^{\frac{m}{16}\|x\|}$. From (2.5), ϕ is bounded by 1 on the set

$$\bigcup_{k \geq k_0} \Big\{ x \in \mathbb{Z}^d \ | \ \|x\| \in [8JL_k, 16JL_k] \Big\} \ .$$

Applying the first inequality in (2.4), we obtain that ϕ is bounded by 1 on $\{\|x\| \geq 8JL_{k_0}\}$. Thus ϕ is bounded, as claimed. □

[1]We may assume that for all k $L_{k+1} \geq (10J)^{100} L_k$.

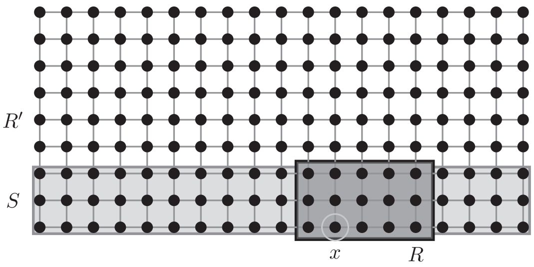

FIGURE 1. Illustration to Lemma 2.8. In this case $d = 2$, $L = 2$ and $L' = 8$; y can be any vertex on $\partial_{\rm in} R$ except for x and the two vertices adjacent to it.

The proof of Proposition 2.6 relies on two lemmata. The first one asserts that the Green function $G_E[H_R]$ in (2.1) can be replaced with $G_E[H_S]$ for $S \supset R$, as long as x is not very close to the boundary of R in S (in particular, it is required that $x \in \partial_{\rm in} R \cap \partial_{\rm in} S$). The following definition will be convenient:

Definition 2.7. Let B be a box. An L-strip $S \subset B$ is a product $S = I'_1 \times \cdots \times I'_d$ of intervals, where $I'_j = I_j$ for $j \neq j_0$, and $\# I'_{j_0} = L$. A set is called a strip if it is an L-strip for some value of L.

Lemma 2.8. *In the setting of Proposition 2.6, let $R \subset R'$ be an E-regular L-rectangle, and let $R \subset S \subset R'$ be a strip (see Figure 1). Then*

$$\forall x, y \in \partial_{in} R \ s.t.\, \mathrm{dist}(x, \{y\} \cup (S \setminus R)) \geq L : \ |G_E[H_S](x,y)| \leq e^{-m(L + \frac{1}{2} L^b)} \ . \tag{2.6}$$

Proof. By assumption (2), the rectangle R' is (E, L)-nonresonant, hence by the resolvent identity

$$\begin{aligned} |G_E[H_S](x,y)| &\leq |G_E[H_R](x,y)| + \sum_{uu' \in \partial R \setminus \partial S} |G_E[H_R](x,u)||G_E[H_S](u',y)| \\ &\leq \exp(-m(L + L^b)) \left[1 + (CL)^{d-1} \exp(\frac{mL^b}{16J}) \right] \\ &\leq \exp(-m(L + \frac{1}{2} L^b)) \end{aligned}$$

if L is sufficiently large, $L \geq L_*(m, b, J, d)$. $\square$

Lemma 2.9. *In the setting of Proposition 2.6, suppose $B \subset R'$ is a box. Let $x, y \in \partial_{in} B$, and let $S \subset B$ be an L-strip such that $x \in \partial_{in} S$ and $y \notin S$. Construct an L-rectangle $R \subset S$ as in Figure 2, left, so that x is the centre of a large face of R (if x is close to the boundary of S, align R with the boundary, as in Figure 2, right). Then*

(1) *if R is regular, then*

$$|G_E[H_B](x,y)| \leq e^{-m(L + \frac{1}{3} L^b)} \max_{vv' \in \partial S \setminus \partial B} |G_E[H_{B \setminus S}](v', y)| \ ;$$

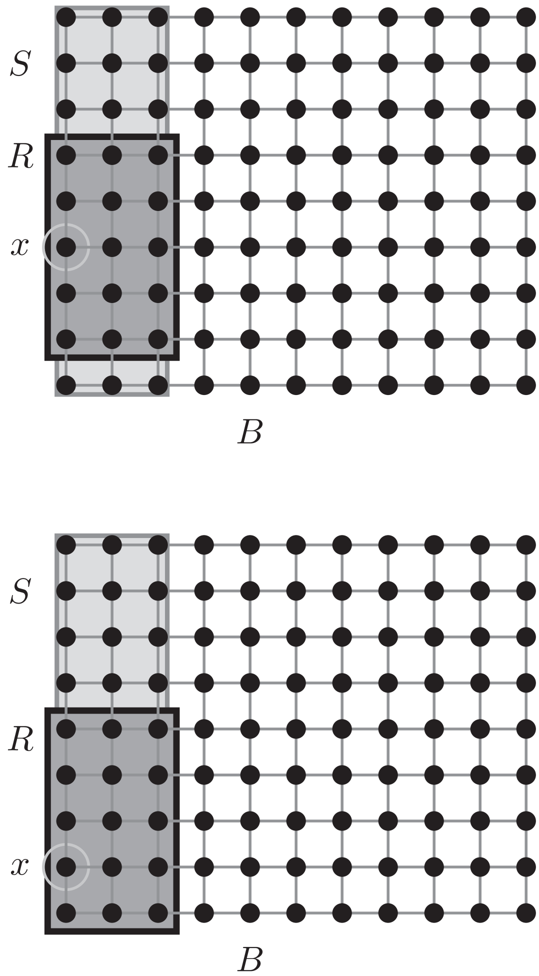

FIGURE 2. Illustration to Lemma 2.9: $d = 2$, $L = 3$. Note that the strip S could also be horizontal.

(2) *if R is singular, then*

$$|G_E[H_B](x,y)| \le e^{+\frac{mL^b}{8J}} \max_{vv' \in \partial S \setminus \partial B} |G_E[H_{B \setminus S}](v',y)| \ .$$

Proof. If R is regular, by the resolvent identity,

$$\begin{aligned} |G_E[H_B](x,y)| &\le \sum_{uu' \in \partial R \setminus \partial B} |G_E[H_B](x,u)||G_E[H_{B\setminus R}](u',y)| \\ &\le \sum_{uu' \in \partial R \setminus \partial B} \sum_{vv' \in \partial S \setminus \partial B} |G_E[H_B](x,u)||G_E[H_{B\setminus R}](u',v)||G_E[H_{B\setminus S}](v',y)| \ . \end{aligned}$$

According to Lemma 2.8, $|G_E[H_B](x,u)| \leq e^{-m(L+\frac{1}{2}L^b)}$, hence

$$|G_E[H_B](x,y)| \leq (2L)^{\nu-1}(2L')^{\nu} e^{-m(L+\frac{1}{2}L^b)} e^{\frac{mL^b}{8J}} \max_{vv' \in \partial S \setminus \partial B} |G_E[H_{B\setminus S}](v',y)|$$
$$\leq e^{-m(L+\frac{1}{3}L^b)} \max_{vv' \in \partial S \setminus \partial B} |G_E[H_{B\setminus S}](v',y)| \ .$$

If R is singular, we argue similarly, starting from the estimate

$$|G_E[H_B](x,y)| \leq \sum_{vv' \in \partial S \setminus \partial B} |G_E[H_B](x,v)||G_E[H_{B\setminus S}](v',y)| \ .$$

$\square$

Proof of Proposition 2.6. Suppose $x, y \in \partial_{\mathrm{in}} R'$, $\|x-y\| \geq L'$. Iterating Lemma 2.9, we obtain

$$|G_E[H_{R'}](x,y)| \leq e^{\frac{mL^b}{16J}} e^{-m(L+\frac{1}{3}L^b)(\frac{L'}{L}-J)} e^{\frac{mL^b}{8J}}$$
$$\leq \exp\left[m\left\{ -L' + L^b\left(\frac{1}{5J} + \frac{1}{3}J\right) - \frac{1}{3}L'L^{b-1} + JL \right\}\right] \tag{2.7}$$
$$\leq \exp\left[m(-L' - \frac{1}{3}L'L^{b-1} + 2JL)\right] \ .$$

If $L' \geq 100JL^{2-b}$, then

$$\frac{1}{3}L^{b-1}L' \geq 2JL + L'^b \ ,$$

hence

$$(2.7) \leq \exp(-m(L' + L'^b)) \ .$$

For arbitrary L' and $x, y \in R'$ with $\|x-y\| \geq 4JL$, a similar argument yields

$$|G_E[H_{R'}](x,y)| \leq e^{-\frac{m}{2}\|x-y\|} \ .$$

$\square$

2.2. **Wegner estimate, and Proof of Proposition 1.4.** Let $H(\omega, \theta; g)$ be an operator of the form

$$(H(\omega,\theta;g)f)(x) = \sum_{\|y-x\|=1} f(y) + gv(T^x\omega, \theta)f(x) \ . \tag{2.8}$$

We recall our basic assumptions:

$$\textbf{(UPA)}_A \qquad \inf_{\omega} \min_{0<\|x\|\leq L} \mathrm{dist}(T^x\omega, \omega) \geq cL^{-A} \tag{2.9}$$

$$\textbf{(LIB)}_\eta \qquad p_\omega(t \mid \Omega \setminus Q_\epsilon(\omega)) \leq \exp(C\epsilon^{-\eta}) \ , \quad \epsilon \in (0, 1/2] \tag{2.10}$$

$$\textbf{(NET)}_\nu \qquad \min \#(\epsilon\text{-net in } \Omega) \leq (C/\epsilon)^\nu \ , \quad \epsilon \in (0,1] \tag{2.11}$$

$$\textbf{(UHöl)}_\kappa \qquad \lim_{R\to\infty} \mathbb{P}^\Theta(\mathfrak{H}_R) = 1 \ , \tag{2.12}$$

where $\mathfrak{H}_R$ is the collection of $\theta \in \Theta$ such that $\|v(\cdot,\theta)\|_\infty \leq R$ and $v(\cdot,\theta)$ is uniformly κ-Hölder with constant R:

$$\sup_{\omega} |v(\omega,\theta)| + \sup_{\omega' \neq \omega} \frac{|v(\omega',\theta) - v(\omega,\theta)|}{\mathrm{dist}(\omega',\omega)^\kappa} \leq R \ . \tag{2.13}$$

Proposition 2.10. *Assume that* $\mathbf{(UPA)}_A$, $\mathbf{(LIB)}_\eta$, $\mathbf{(NET)}_\nu$ *and* $\mathbf{(UH\ddot{o}l)}_\kappa$ *hold with* $A\eta < 1$. *Let*

$$m = 16 \ , \quad J = \min(\mathbb{Z} \cap (\frac{\nu}{\kappa} + 1, \infty)) \ ,$$

and choose $b \in (0,1)$ *and* $\gamma \in (2-b, \infty)$ *so that* $A\eta < b/\gamma^2$. *Then there exist two measurable functions* $L_{\min}(\omega, \theta)$ *and* $g_{\min}(\omega, \theta)$ *that are* Θ*-almost-everywhere finite for each* $\omega \in \Omega$, *such that for* $L_0 \geq L_{\min}$, $g \geq g_{\min}$ *the assumptions (1)–(2) of Proposition 2.4 hold for the operator* $H(\omega, \theta; g)$.

The proof is based on the following lemma. For $r > 0$, $E \in \mathbb{R}$, $\omega \in \Omega$ and $s_1, \cdots, s_k \subset \mathbb{Z}^d$, define the following events in Θ:

$$\text{Reson}_{L,r}(s_1, \cdots, s_k; \omega; E) = \left\{ \forall j = 1, \cdots, k \ \|G_E[H_{s_j}(\omega, \theta; g)]\| > \frac{e^{L^r}}{g} \right\} \tag{2.14}$$

$$\text{Reson}_{L,r}(s_1, \cdots, s_k; \omega) = \bigcup_{E \in \mathbb{R}} \text{Reson}_{L,r}(s_1, \cdots, s_k; \omega; E) \tag{2.15}$$

$$\text{Reson}_{L,r}(s_1, \cdots, s_k) = \bigcup_{\omega \in \Omega} \text{Reson}_{L,r}(s_1, \cdots, s_k; \omega) \tag{2.16}$$

Lemma 2.11. *Assume that* $\mathbf{(UPA)}_A$, $\mathbf{(LIB)}_\eta$, $\mathbf{(NET)}_\nu$ *hold with* $A\eta < 1$. *Let* m, b, γ, J *be as in Proposition 2.10, and let* $r > A\eta$, $R \geq 1$.[2] *Then*

(1) *for* $k \geq 2$,

$$\sup_{\omega \in \Omega} \sup_{s_1, \cdots, s_k} \mathbb{P}^\Theta(\text{Reson}_{L,r}(s_1, \cdots, s_k; \omega) \cap \mathfrak{H}_R) \leq R \exp(-(k-1)L^r - o(L^r)) \ ;$$

(2) *for* $k > \frac{\nu}{\kappa} + 1$,

$$\sup_{s_1, \cdots, s_k} \mathbb{P}^\Theta(\text{Reson}_{L,r}(s_1, \cdots, s_k) \cap \mathfrak{H}_R) \leq R^{\frac{\nu}{\kappa}+1} \exp\left(-\left(k - \frac{\nu}{\kappa} - 1\right)L^r - o(L^r)\right) \ ,$$

where the supremum in the first formula and the interior one in the second formula are over k*-tuples of pairwise disjoint subsets of* $[-L, L]^d$.

Proof. Fix $\omega \in \Omega$ and $E \in \mathbb{R}$. From $\mathbf{(UPA)}_A$ and $\mathbf{(LIB)}_\eta$, the joint probability density (in Θ) of $(V(x;\omega))_{x \in B}$, $B \subset [-L, L]^d$, is bounded by

$$\left(\frac{\exp(C(cL^{-A})^{-\eta})}{g} \right)^{\#B} ,$$

therefore by the usual Wegner argument [3, 44], we obtain that for $M > 0$

$$\begin{aligned} &\mathbb{P}^\Theta \left\{ \forall j = 1, \cdots, k \ \|G_E[H_{s_j}(\omega, \theta)]\| > M \right\} \\ &\leq \left(\frac{\exp(C(cL^{-A})^{-\eta})}{gM} \right)^k \prod_{j=1}^k \#s_j \leq \left(\frac{(3L)^d \exp(C_1 L^{A\eta})}{gM} \right)^k . \end{aligned} \tag{2.17}$$

Let $M = \frac{1}{4g} \exp(L^r)$; then

$$\text{RHS of (2.17)} \leq \left[4(3L)^d \exp(C_1 L^{A\eta} - L^r) \right]^k \leq \exp(-kL^r + o(L^r)) \ ;$$

[2] Eventually, r will be taken to be slightly greater than $A\eta$, however, no upper bound is formally required in the current lemma. R will eventually play the same rôle as in (2.13).

here and in the sequel the implicit constants are uniform in s_j and ω. Let $\mathcal{N}_\Omega$ be an $(4gMR)^{-1/\kappa}$-net in Ω, and $\mathcal{N}_\mathbb{R}$ – a $(4M)^{-1}$-net in $[-10dgR, 10dgR]$, chosen so that

$$\#\mathcal{N}_\Omega \le (CgMR)^{\nu/\kappa} , \quad \#\mathcal{N}_\mathbb{R} \le CdgMR .$$

Then

$$\begin{aligned}&\mathbb{P}^\Theta\left\{\exists E \in \mathcal{N}_R :\ \forall j = 1, \cdots, k\ \|G_E[H_{s_j}(\omega,\theta)]\| \ge M\right\}\\ &\quad\le CdgMR\exp(-kL^r + o(L^r)) \le R\exp(-(k-1)L^r + o(L^r))\end{aligned} \tag{2.18}$$

for any $\omega \in \Omega$, and

$$\begin{aligned}&\mathbb{P}^\Theta\left\{\exists E \in \mathcal{N}_R,\ \omega \in \mathcal{N}_\Omega :\ \forall j = 1, \cdots, k\ \|G_E[H_{s_j}(\omega,\theta)]\| \ge M\right\}\\ &\quad\le (CgMR)^{\frac{\nu}{\kappa}} R\exp(-(k-1)L^r + o(L^r))\\ &\quad\le R^{\frac{\nu}{\kappa}+1}\exp(-(k - \frac{\nu}{\kappa} - 1)L^r + o(L^r)) .\end{aligned} \tag{2.19}$$

If $\|G_E[H_s(\omega,\theta)]\| \le M$, $\theta \in \mathfrak{H}_R$, $|E' - E| \le \frac{1}{4M}$, and $\mathrm{dist}(\omega',\omega) \le (4gMR)^{-1/\nu}$, then

$$\|G_{E'}[H_s(\omega',\theta)]\| \le 2M . \tag{2.20}$$

Also note that on $\mathfrak{H}_R$ the bound (2.20) holds for all $|E| \ge 10dgR$: indeed, such energies are at distance ≥ 1 from the spectrum of H, Therefore (2.18) and (2.19) imply the first and second assertions of the lemma, respectively. □

Proof of Proposition 2.10. Fix $\omega_0 \in \Omega$. Denote by $\mathrm{Bad}_L(\omega_0)$ the event (in Θ-space) that either there exist $E \in \mathbb{R}$ and $\omega \in \Omega$ such that (E, L)-resonant $\lfloor L^\gamma\rfloor$-rectangles are not J-sparse in

$$B_L = [-\lfloor\lfloor L^\gamma\rfloor^\gamma\rfloor, \lfloor\lfloor L^\gamma\rfloor^\gamma\rfloor]^d ,$$

for $H(\omega,\theta)$, or there exists E such that (E, L)-resonant $\lfloor L^\gamma\rfloor$-rectangles are not 2-sparse in B_L for $H(\omega_0,\theta)$. According to Lemma 2.11 applied with an arbitrary $r \in (A\eta, b/\gamma^2)$ and with $\lfloor\lfloor L^\gamma\rfloor^\gamma\rfloor$ in place of L,

$$\mathbb{P}(\mathrm{Bad_L} \cap \mathfrak{H}_R) \le R^{\frac{\nu}{\kappa}+1}\exp(-cL^r + o(L^r)) ,$$

where $c = \min(J - \frac{\nu}{\kappa} - 1, 1) > 0$. Thus for every $R \ge 1$

$$\mathbb{P}(\limsup_{L\to\infty} \mathrm{Bad}_L \cap \mathfrak{H}_R) = 0 .$$

Combining this with $\textbf{(UHöl)}_\kappa$, we obtain that almost every θ lies in $\mathfrak{H}_R \setminus \mathrm{Bad}_L$ for all sufficiently large R and L (i.e. $R \ge R_{\min}(\theta)$ and $L \ge L_{\min}(\theta)$).

Then for $L_0 \ge L_{\min}(\theta)$ each $H(\omega,\theta)$ satisfies that for all $k \ge 0$ (E, L_k)-resonant L_{k+1}-rectangles are J-sparse in any L_{k+2}-rectangle. Indeed, the restriction of $H(\omega,\theta)$ to any L_{k+2}-rectangle coincides with the restriction of $H(\omega',\theta)$ to $[-L_{k+2}, L_{k+2}]^{d-1} \times [1, L_{k+2}]$ for an appropriately chosen ω'. Also, for $H(\omega_0,\theta)$, (E, L_k)-resonant L_{k+1}-rectangles and 2-sparse in $[-L_{k+2}, L_{k+2}]^d$. Thus the first half of assumption (1) of Proposition 2.4 holds.

Next, let $g \ge 10^{10}de^{L^r}$. For any L_1-rectangle R' and any disjoint L_0-rectangles $R_1, \cdots, R_J \subset R'$, there exists $j \in \{1, \cdots, J\}$ such that

$$\|G_E[H_{R_j}]\| \le \frac{\exp(L^r)}{g} , \quad \text{i.e.} \quad \mathrm{dist}(E, \sigma(H_{R_j})) \ge \frac{g}{\exp(L^r)} \ge 10^{10}d ,$$

therefore R_j is E-regular by the Combes–Thomas bound [3]. Hence also asumption (2) of Proposition 2.4 holds. □

Proof of Proposition 1.4. For every ω and almost every θ there exist $L_{\min}$ and $g_{\min}$ such that the assumptions of Proposition 2.4 hold for $L \geq L_{\min}$ and $g \geq g_{\min}$. Denote by $\mathrm{Assum}_{g,L}$ the set of (ω, θ) for which these assumptions hold with the given values g and L. Then for any $\delta > 0$ there exist L_δ and g_δ such that for $L \geq L_\delta$ and $g \geq g_\delta$

$$\mathbb{P}_{\Omega\times\Theta}(\mathrm{Assum}_{g,L}) \geq 1 - \delta \ .$$

Denote

$$\mathrm{Assum}^\theta_{g,L} = \{\omega : (\omega, \theta) \in \mathrm{Assum}_{g,L}\} \ .$$

Then

$$\mathbb{P}_\Theta\left(\left\{\theta :\ P_\Omega(\mathrm{Assum}^\theta_{g,L}) \leq \frac{1}{2}\right\}\right) \leq 2\delta \ .$$

If θ does not lie in this set, then by ergodicity there exists a shift of the operator $H(\omega, \theta)$ for which the the assumptions of Proposition 2.4 hold. Invoking Proposition 2.4, we obtain the result. □

3. Interpolation of Gaussian processes

The general strategy is as follows. A lemma of [27], which we reproduce in Section 3.1, reduces the proof of Proposition 1.6 to the construction of a compactly supported function with prescribed decay of the Fourier transform. In Section 3.2 we construct such a function by adjusting the arguments of [34, 40, 41].

3.1. A formula of Karhunen.

We use the conventions

$$\hat{g}(\lambda) = \int g(\xi)\exp(-i\langle\xi, \lambda\rangle)d\xi \tag{3.1}$$

$$\check{h}(\xi) = \int h(\lambda)\exp(i\langle\xi, \lambda\rangle)\frac{d\lambda}{(2\pi)^\nu} \tag{3.2}$$

for the Fourier transform of $g : \mathbb{R}^\nu \to \mathbb{C}$ and its inverse, and

$$\hat{g}(\ell) = \int_{\mathbb{T}^\nu} g(\omega)\exp(-i\langle\omega, \ell\rangle)d\xi \tag{3.3}$$

$$\check{h}(\omega) = \sum_{\ell\in 2\pi\mathbb{Z}^\nu} h(\ell)\exp(i\langle\omega, \ell\rangle) \tag{3.4}$$

for the Fourier transform of $g : \mathbb{T}^\nu \to \mathbb{C}$ and its inverse. With these conventions,

$$\int_{\mathbb{R}^\nu} |\hat{g}(\lambda)|^2 d\lambda = (2\pi)^\nu \int_{\mathbb{R}^\nu} |g(\xi)|^2 d\xi \qquad (\mathbb{R}^\nu) \tag{3.5}$$

$$\sum_{\ell\in 2\pi\mathbb{Z}^\nu} |\hat{g}(\ell)|^2 = \int_{\mathbb{T}^\nu} |g(\xi)|^2 d\xi \qquad (\mathbb{T}^\nu)\ . \tag{3.6}$$

The following lemma goes back to the work of [27] (see further [19, §4.13, Test 2]).

Lemma 3.1 (Karhunen)**.** *For $v(\omega)$ as in (1.2),*

$$\begin{aligned}\mathbf{V}(\epsilon) &\stackrel{def}{=} \operatorname{Var}\left(v(\omega) \,\middle|\, \{v(\omega') \,:\, \|\omega' - \omega\| \geq \epsilon\}\right) \\ &= \sup\left\{ \frac{|g(0)|^2}{\sum_\ell |\hat{g}(\ell)|^2 W(\ell)} \mid \operatorname{supp} g \subset \{\|\omega\| < \epsilon\} \right\} .\end{aligned}$$

Proof. We prove the inequality "$\geq$", as this is the direction we use in the sequel. Let $\tilde{v}$ be an independent copy of v, and let

$$X(\omega) = \frac{v(\omega) + \tilde{v}(\omega)}{\sqrt{2}} = \sum_{\ell \in 2\pi\mathbb{Z}^\nu} \frac{G_\ell e^{i\langle \omega, \ell \rangle}}{\sqrt{W(\ell)}} ,$$

where G_ℓ are independent standard *complex* Gaussian variables. It suffices to prove the equality for $\mathbf{V}(\epsilon)$ defined for X in place of v. We start from the relation

$$\mathbf{V}(\epsilon) = \inf\left\{ \mathbb{E}\left| X(0) - \int X(\omega)\rho(\omega) d\omega \right|^2 \mid \rho \in L_2(\mathbb{T}^\nu) \ , \ \operatorname{supp} \rho \subset \{\|\xi\| \geq \epsilon\} \right\} .$$

Rewrite

$$\begin{aligned}&\mathbb{E}\left| X(0) - \int X(\omega)\rho(\omega) d\omega \right|^2 \\ &\quad= \mathbb{E}\left| \sum_{\ell \in 2\pi\mathbb{Z}^d} \frac{G_\ell}{\sqrt{W(\ell)}} \left(1 - \int e^{i\langle\omega,\ell\rangle} \rho(\omega) d\omega\right) \right|^2 \\ &\quad= \mathbb{E}\left| \sum_{\ell \in 2\pi\mathbb{Z}^d} \frac{G_\ell}{\sqrt{W(\ell)}} (1 - \overline{\hat{\rho}(\ell)}) \right|^2 = \sum_{\ell \in 2\pi\mathbb{Z}^d} \frac{|1 - \overline{\hat{\rho}(\ell)}|^2}{W(\ell)} .\end{aligned}$$

For an arbitrary ρ supported in $\{\|\omega\| \geq \epsilon\}$ and an arbitrary g supported in $\{\|\omega\| \leq \epsilon\}$,

$$g(0) = g(0) - \int g(\omega)\rho(\omega) d\omega = \sum \hat{g}(\ell)(1 - \overline{\hat{\rho}(\ell)}) ,$$

whence by Cauchy–Schwarz

$$|g(0)|^2 \leq \left(\sum |\hat{g}(\ell)|^2 W(\ell) \right) \times \left(\sum \frac{|1 - \overline{\hat{\rho}(\ell)}|^2}{W(\ell)} \right) .$$

Thus

$$\mathbf{V}(\epsilon) \geq \frac{|g(0)|^2}{\sum_\ell |\hat{g}(\ell)|^2 W(\ell)} . \qquad \square$$

3.2. **Functions with prescribed Fourier decay.** The following proposition is a quantitative version of a result proved in [40] and [34] in dimension $\nu = 1$, and in [41] in arbitrary dimension. The method of convolutions used in the proof was applied for similar purpose already in [34], and for the proof of necessity in the Denjoy–Carleman theorem – in [35] (where an earlier unpublished work of Bray is quoted) and in [4]; see further [24, §1.3 and Notes] and [33, §25].

Proposition 3.2. *Let $M : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that*

$$M(0) = 1 \ , \quad \int^\infty \frac{\log M(t)}{t^2} dt < \infty \ .$$

Then for any $\nu \geq 1$ and $\epsilon \in (0,1]$ there exists $g : \mathbb{R}^\nu \to \mathbb{R}_+$ such that

$$\operatorname{supp} g \in [-\epsilon, \epsilon]^\nu \ , \qquad g(0) = \max g \ , \quad \hat{g}(0) = 1 \ , \tag{3.7}$$

$$|\hat{g}(\lambda)| \leq \frac{eM(S^{-1}(\epsilon/e))}{M(\|\lambda\|)} \ , \quad \text{where} \quad S(t) = \int_t^\infty \frac{\log M(\tau)}{\tau^2} d\tau \ . \tag{3.8}$$

Proof. Let $u(\xi) = 2^{-\nu} \mathbb{1}_{[-1,1]^\nu}(\xi)$, so that $\hat{u}(\lambda) = \prod_{r=1}^\nu \frac{\sin \lambda_r}{\lambda_r}$. Then

$$|\hat{u}(\lambda)| \leq \min(1, \|\lambda\|^{-1}) \ . \tag{3.9}$$

We may assume that M is continuous. Let

$$R_j = \min \left\{ t \geq 0 \ | \ M(t) = e^j \right\} \ ,$$

and choose k_0 so that

$$S(R_{k_0}) \leq \frac{\epsilon}{e} \ , \ S(R_{k_0 - 1}) > \frac{\epsilon}{e} \ .$$

Define

$$\hat{g}(\lambda) = \prod_{j=k_0}^\infty \hat{u}(\frac{e\lambda}{R_j}) \ .$$

Then $\max \hat{g} = g(0)$ and $\hat{g}(0) = 1$, and

$$\operatorname{supp} g \subset [-\sum_{j=k_0}^\infty \frac{e}{R_j}, \sum_{j=k_0}^\infty \frac{e}{R_j}] \subset [-\epsilon, \epsilon]^\nu \ ,$$

since

$$\begin{aligned}
\sum_{j=k_0}^\infty \frac{1}{R_j} &= \int_{R_{k_0}}^\infty \frac{dt}{t^2} \# \{k_0 \leq j \leq t\} \\
&\leq \sum_{j \geq k_0} \int_{R_j}^{R_{j+1}} \frac{dt}{t^2} (j - k_0 + 1)_+ \\
&\leq \sum_{j \geq k_0} \int_{R_j}^{R_{j+1}} \frac{\log M(t)}{t^2} dt = S(R_{k_0}) \leq \frac{\epsilon}{e} \ .
\end{aligned}$$

This proves (3.7), and we turn to the proof of (3.8). By (3.9), we have for $R_k \leq \|\lambda\| < R_{k+1}$:

$$\begin{aligned}
|\hat{g}(\lambda)| &\leq \prod_{j \geq k_0} \min(1, \frac{R_j}{e\|\lambda\|}) \\
&\leq \prod_{j=k_0}^k \frac{1}{e} = \exp(-(k - k_0 + 1)_+) \ .
\end{aligned}$$

On the other hand,

$$M(\|\lambda\|) \leq M(R_{k+1}) \leq \exp(k+1) \ .$$

Hence

$$|\hat{g}(\lambda)| \leq e^{k_0}/M(\|\lambda\|) \leq eM(S^{-1}(\epsilon/e))/M(\|\lambda\|) \ ,$$

as claimed. □

3.3. **Proof of Proposition 1.6.** We apply Proposition 3.2 with $M_1(t) = \sqrt{M(t)}$, and $S_1(t) = \frac{1}{2}S(t)$. The function g thus obtained satisfies

$$|\hat{g}(\ell)| \leq \frac{eM_1(S_1^{-1}(\epsilon/e))}{M_1(\|\ell\|)} = \frac{e\sqrt{M(S^{-1}(\frac{2}{e}\epsilon)}}{\sqrt{M(\|\ell\|)}} \ ,$$

whence

$$\sum |\hat{g}(\ell)|^2 W(\ell) \leq K \max |\hat{g}(\ell)|^2 M(\ell) \leq e^2 K M\left(S^{-1}\left(\frac{2}{e}\epsilon\right)\right) \ .$$

On the other hand,

$$|g(0)|^2 = \max_{\omega} |g(\omega)|^2 \geq \left[\frac{1}{(2\epsilon)^\nu}\int g(\omega)d\omega\right]^2 = \frac{1}{(2\epsilon)^{2\nu}} \ .$$

Thus by Lemma 3.1

$$\mathbf{V}(\epsilon) \geq \frac{1}{e^2 2^{2\nu} K \epsilon^{2\nu} M(S^{-1}(\frac{2}{e}\epsilon))} \ ,$$

as claimed. □

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